Mathematical aspects of the combinatorial game "Mahjong"

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Joint work with

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- It is a game of skill, strategy, calculation, and some luck.
- There has been research suggesting that Mahjong is a good cognitive game with positive impact for patients with Alzhemier's disease.
- We will explore some mathematical aspects of the Mahjong game.

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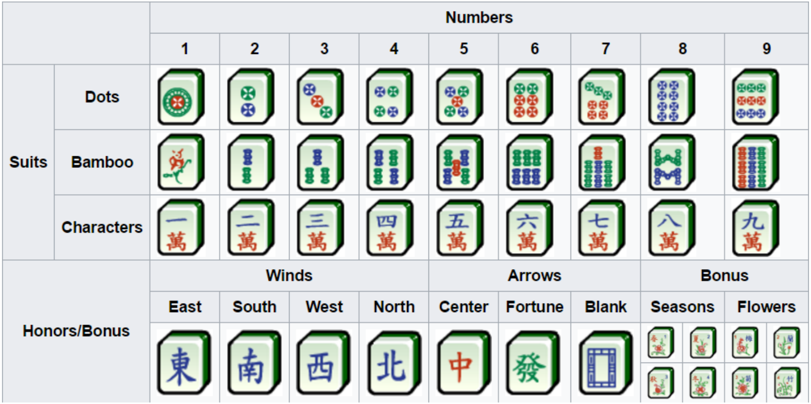

		Numbers								
		1	2	3	4	5	6	7	8	9
Suits	Dots									
	Bamboo									
	Characters	一萬	二萬	三萬	四萬	五萬	六萬	七萬	八萬	九萬
Honors/Bonus		Winds				Arrows			Bonus	
		East	South	West	North	Center	Fortune	Blank	Seasons	Flowers
		東	南	西	北	中	發			

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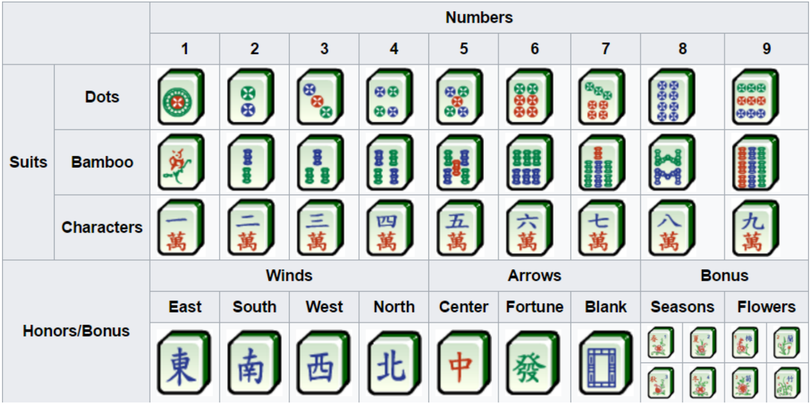

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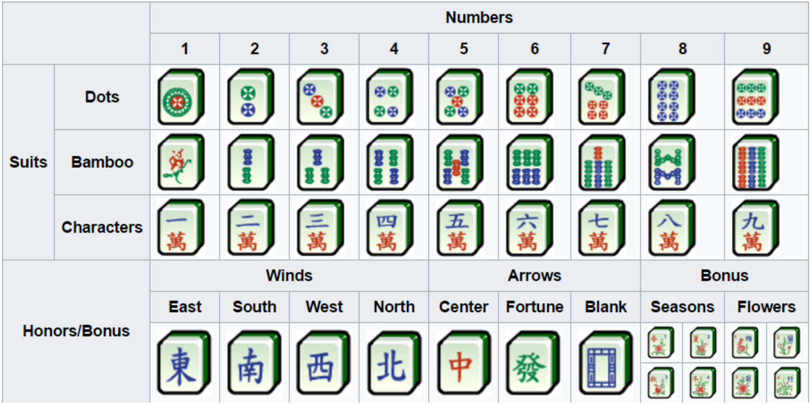

with 36 tiles of dot type, 36 tiles of bamboo type, 36 tiles of character type, and some special tiles including 4 different flowers, 4 different seasons, and 4 copies of each of the other tiles.

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- This hand is special because any additional dot tile would lead to a winning hand.

- For example, if we draw a one dot tile, then we get a winning hand:

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- However, there is no known mathematical proof of this folklore.
- In fact, I believe that this is the only way that one can use nine different tiles to form a winning pattern with a hand of 13 tiles.

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- The other cases are difficult. We solve the problems by computer programming with some basic combinatorial theory.
- We focus on Mahjong hands of 13 tiles chosen from the 36 dot tiles to study the questions of “Nine Gates”, “Eight Gates”, etc.

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Mathematical theory

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- Use
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- Denote a hand by a "product" of 13 terms such as
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- In general, a 13-dot hand is represented as
$$X_1^{n_1} \cdots X_9^{n_9}, \quad n_1 + \cdots + n_9 = 13.$$

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- In particular, the probability of getting the hand of nine gates is

$$\binom{36}{13}^{-1} \binom{4}{1}^9 = \frac{262144}{2310789600} = 0.00011344347 = 0.011344347\%.$$

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- As calculated before, we have $\alpha_{13} = 93600$.

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  such that $0 \le n_j \le 4$ for every j.
- We modify the program for the remaining pieces easily to check what are needed to form a winning pattern for a reduced hand after some "pungs" or "chows" were performed in a game.

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- For each of the tiles that appeared at least twice in the hand, we take two of them out as the pair.
- If the remaining 12-tile hand $\{j_1, \ldots, j_{12}\}$ can be divided into four sets of pungs and chows, then this is a winning hand.

Proposition

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- But then we can rearrange the twelve tiles as three sets of chows into three set of pungs $\{j_1, j_1, j_1\}, \{j_1 + 1, j_1 + 1, j_1 + 1\}, \{j_1 + 2, j_1 + 2, j_1 + 2\}$ together with the remaining set of pung or chow.

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So, we can remove $\{j_1, j_2, j_3\}$ from $\{j_1, \ldots, j_{12}\}$, if the three smallest number in the remaining set are the same.

Else, we will extract a set of chow and proceed in a similar manner.

Computational Results and Implications

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- The comment applies to the hand where the k-dot tile cannot be added to form a winning hand for $k = 3, 6, 7$.
- In particular, when $k = 3$ there are two such "Eight Gates" hand. The same is true for $k = 7$.

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- There are 53530 hands which cannot win with any additional piece, with a combined probability 52.4409% chance of drawing.

- There are too many hands corresponding to "Seven Gates", "Six Gates", "Five Gates", etc. One may see the spread sheet at

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- Our results show that out of the $\binom{9}{3} = 84$ possible choices of $\{X_i, X_j, X_k\}$ one can get "Three Gates" hands with these sets of winning tiles with the following 11 exceptions:

$$\{X_1, X_2, X_9\}, \{X_1, X_3, X_8\}, \{X_1, X_5, X_7\}, \{X_1, X_5, X_9\}, \{X_1, X_6, X_8\}, \{X_1, X_8, X_9\},$$

$$\{X_2, X_4, X_8\}, \{X_2, X_4, X_9\}, \{X_2, X_6, X_8\}, \{X_2, X_7, X_9\}, \{X_3, X_5, X_9\}.$$

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- This result is higher than many Mahjong players would expect.

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[1, 1, 2, 2, 2, 3, 3, 3, 4, 5, 6, 7, 8, 9, 9, 9], [1, 1, 1, 2, 3, 4, 5, 6, 7, 7, 7, 8, 8, 8, 9, 9],

[1, 1, 1, 2, 3, 4, 5, 6, 6, 7, 7, 8, 8, 9, 9, 9], [1, 1, 1, 2, 3, 4, 5, 6, 6, 6, 7, 7, 7, 8, 8, 8],

[1, 1, 1, 2, 3, 4, 5, 5, 6, 6, 7, 7, 8, 9, 9, 9], [1, 1, 1, 2, 3, 4, 4, 5, 5, 6, 6, 7, 8, 9, 9, 9],

[1, 1, 1, 2, 3, 3, 4, 4, 5, 5, 6, 7, 8, 9, 9, 9], [1, 1, 1, 2, 2, 3, 3, 4, 4, 5, 6, 7, 8, 9, 9, 9],

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- There are 94 hands of eight gates.
- If one pick 17 tiles out of the 36 dot tiles, the probability of winning is:

$$15.031441172286243\%$$

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- It would involve psychology, game theory, artificial intelligence, machine learning, etc.
- I also extend the study to Quantum mahjong.

Hope to tell you more next time!

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Thank you for your attention!