Math 408 Advanced Linear Algebra

The scores of the best 7 questions will go to the final examination grade.
The rest will be extra homework credits. Good luck!

1. (a) Use Gershgorin Theorem to conclude that \(\begin{pmatrix} 1 & 1/2 & 1 \\ i & 4 & 0 \\ 0 & i/2 & 7 \end{pmatrix}\) has 3 distinct eigenvalues.

   (b) For \(B = (b_{ij})\), let \(G_1(B) = \{\mu \in \mathbb{C} : |\mu - b_{ii}| \leq \sum_{j \neq i} |b_{ij}|\}\). Construct an example of \(B\) so that there is no eigenvalue in \(G_1(B)\).

2. Let \(A \in M_{n,p}\) and \(B \in M_{n,q}\). Show that

\[
\text{rank } ([A|B]) = \text{rank } (A) + \text{rank } (B) - d,
\]

where \(d\) is the dimension of \(\text{Col}(A) \cap \text{Col}(B)\), the intersection of the column space of \(A\) and the column space of \(B\).

Hint: Let \(\{v_1, \ldots, v_d\}\) be a basis for \(\text{Col}(A) \cap \text{Col}(B)\). Form bases for \(\text{Col}(A), \text{Col}(B)\) containing \(\{v_1, \ldots, v_d\}\), and ....

3. Let \(A \in M_n\) be a normal matrix.

   (a) If \(B \in M_m\) is normal, show that \(A \otimes B\) is normal.

   (b) If \(m > 1\) and \(C \in M_m\) is not normal, show that \(A \otimes C\) is not normal.

4. Prove or disprove the following.

   (a) A principal submatrix of a Hermitian matrix \(A \in M_n\) is Hermitian.

   (b) A principal submatrix of a normal matrix is normal.

5. Let \(A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n\) such that \(A_{11} \in M_k\) is invertible.

   (a) Compute \(RA\) for \(R = \begin{pmatrix} I_k & 0 \\ -A_{21}A_{11}^{-1} & I_{n-k} \end{pmatrix}\).

   (b) Show that \(A\) is invertible if and only if \(C = A_{22} - A_{21}A_{11}^{-1}A_{12}\) is invertible, and \(A^{-1}\) has the form \(\begin{pmatrix} * & * \\ * & C^{-1} \end{pmatrix}\).

   Hint: If \(T = RA = \begin{pmatrix} T_{11} & * \\ 0 & T_{22} \end{pmatrix}\) is invertible, then show that \(T^{-1}\) has the form \(\begin{pmatrix} T_{11}^{-1} & * \\ 0 & T_{22}^{-1} \end{pmatrix}\).

6. Let \(A \in M_n\) be Hermitian. Show that if \(A \in M_n\) is positive semidefinite if and only if all principal minors of \(A\) are nonnegative.
7. (a) Show that if $A \in M_n$ is normal, then $r(A) = w(A) = s_1(A)$.
    (b) Construct a non-normal $B \in M_n$ such that $r(B) = w(B) = s_1(B)$.

8. Let $(a_1, b_1), \ldots, (a_n, b_n) \in \mathbb{C}^{1 \times 2}$ be such that $a_1, \ldots, a_n$ are distinct.
    (a) Suppose $f(z) = f_{n-1}z^{n-1} + f_{n-2}z^{n-2} + \cdots + f_0$ satisfies $f(a_i) = b_i$ for $i = 1, \ldots, n$. Show that
        $$V(a_1, \ldots, a_n)(f_0, \ldots, f_{n-1})^t = (b_1, \ldots, b_n)^t,$$
        where the $i$th row of $V(a_1, \ldots, a_n) \in M_n$ is $(1 \ a_1^2 \ a_1^{n-1})$ for $i = 1, \ldots, n$.
    (b) Show that $\det(V(a_1, \ldots, a_n)) = \prod_{1 \leq i < j \leq n}(a_j - a_i)$; thus $V(a_1, \ldots, a_n)$ is always invertible.
        Hint: You may use induction, or show that $(a_j - a_i)$ is always a factor of $\det(V(a_1, \ldots, a_n))$ if we regard $\det(V(a_1, \ldots, a_n))$ as a multinomial of $(a_1, \ldots, a_n)$. Then conclude that the expansion of $\det(V(a_1, \ldots, a_n))$ is the same as the expansion of $\prod_{1 \leq i < j \leq n}(a_j - a_i)$.

Remark The matrix $V(a_1, \ldots, a_n)$ is known as the Vandetmonde matrix.

9. Let $A \in M_n$.
    (a) Use the fact that $A$ contains all the eigenvalues of $A$ and the convexity of the numerical range to show that $(\text{tr } A)/n \in W(A)$.
    (b) Show that there is a unitary $U$ such that $U^*AU$ has all diagonal entries equal to $(\text{tr } A)/n$.
        [Hint: Induction.]

10. Let $N_n = E_{12} + \cdots + E_{n-1,n} \in M_n$ and
    $$T_n = aI_n + bN_n + cN_n^t = \begin{pmatrix} a & b & \cdots & \cdots & c \\ c & a & b & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ c & a & b \\ c & a \end{pmatrix}.$$
    (a) Show that $\det(T_1) = a, \det(T_2) = a^2 - bc, \det(T_n) = a \det(T_{n-1}) - bc \det(T_{n-2})$.
    (b) If $bc = 0$, show that $\det(T_n) = a^n$.
    (c) Suppose $bc \neq 0$ and $\alpha, \beta$ are the zeros of $f(z) = z^2 - az + bc$.
        (c.1) If $\alpha = \beta$, show that $\det(T_n) = (n + 1)(a/2)^n$.
        (c.2) If $\alpha \neq \beta$, show that $\det(T_n) = c_1\alpha^n + c_2\beta^n$ with $c_1 = \alpha/(\alpha - \beta)$ and $c_2 = -\beta/(\alpha - \beta)$.
        Hint: Suffices to show that the suggested values in (c.1), (c.2) satisfy the conditions in (a).