1. Let \( A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}. \)

(a) Determine \( S \) such that \( S^{-1}AS \) is a direct sum of the Jordan block.

(b) What is minimal polynomial of \( A \)?

(c) Suppose \( f(z) \) is a polynomial. What are the possible Jordan form of \( f(A) \)?

Hint: Suppose \( f(z) = m_A(z)q(z) + r(z) \). Then \( r(z) = a_0z + a_1 \) because ....

So, \( f(A) = r(A) \) has Jordan form ....

2. If \( A \in M_5 \) has distinct eigenvalues \( 1, i \), determine all the possible Jordan forms of \( A \).

Hint: \( \det(zI - A) = (x - 1)^r(x - i)^s \) with \( r, s > 0, r + s = 5 \). So, ...

3. Suppose \( A \in M_5 \) is similar to \( J_2(i) \oplus J_2(1) \oplus J_1(1) \). If \( f(z) \) is a polynomial, what are the possible Jordan form of \( f(A) \).

Hint: Suppose \( f(z) = m_A(z)q(z) + r(z) \). For each Jordan block \( J_k(\lambda) \) determine \( r(J_k(\lambda)) \) depending on whether \( r(\lambda) = 0 \).

4. Suppose \( f(z) \) is a polynomial, and \( A \in M_n \).

(a) If \( Ax = \lambda x \) for a nonzero vector \( x \), show that \( f(A)x = f(\lambda)x \).

(b) Show that an eigenvector of \( f(A) \) may not be an eigenvector of \( A \).

Hint: (a) Show that \( A^k = \lambda^k x \) for \( k = 1, 2, \ldots \). Then consider general \( f(z) \).

5. Suppose \( A \) is \( m \times n \) and \( B \) is \( n \times m \). Then \( AB \) and \( BA \) have the same set of nonzero eigenvalues of the same multiplicities.

Hint: Show that \( \begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_m & A \\ 0 & B \end{pmatrix} \begin{pmatrix} 0_m & 0 \\ A & BA \end{pmatrix} \).

6. Suppose \( f(z) = z^n + a_1z^{n-1} + \cdots + a_n \). Then

\[
A_f = \sum_{j=1}^{n-1} E_{j+1,j} - \sum_{j=1}^{n} a_j E_{1j} = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 \end{pmatrix}
\]

is the companion matrix of \( f \). Here \( \{E_{11}, E_{12}, \ldots, E_{nn}\} \) is the standard basis for \( M_n \).

(a) Show that \( \det(zI - A_f) = f(z) \) by expanding \( \det(zI - A_f) \) using the last row, and induction.

(b) Show that \( f(z) \) is the minimal polynomial of \( A_f \).

Hint: Show that \( A - \lambda_i I \) has rank \( n - 1 \) for each distinct eigenvalue \( \lambda_i \).
7. (Extra Credits) Suppose $A = J_m(\lambda)$ and $x'(s) = Ax(s)$. Show that the system of differential equation has a solution of the form:

$$y_k(s) = q_k(s)e^{s\lambda}, \quad k = 1, \ldots, m,$$

where $q_k(s) = c_{k0} + c_{k1}s + \cdots + c_{m-k,m-k} s^{m-k}$ is a polynomial in $s$ of degree $m - k$.

Hint: The result is true for $k = m$. Then show that it is true for $k = m - 1, m - 2, \ldots$ by backward induction.