1. Let \( A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \).

(a) The only eigenvalue of \( A \) is 3 and the eigenvectors are the elements of 
\[
Nul(A - 3I) = Nul \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}.
\]

Note that \( \dim(Nul(A - 3I)) = 2 \) and thus, there are two jordan blocks. The only possible \( 3 \times 3 \) Jordan canonical form satisfying this is 
\[
J = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}
\]

Next we need to find an invertible \( S = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \in M_3 \) satisfying \( S^{-1}AS = J \), or equivalently \( AS = SJ \). Thus \( Av_1 = 3v_1, (A - 3)v_2 = v_1 \) and \( Av_3 = 3v_3 \). We can take \( v_1 = e_1 \) and \( v_3 = -2e_2 + e_3 \). Note that \( (A - 3)(-2e_1 + e_2) = e_1 \). Thus, we can take \( v_2 = -2e_1 + e_2 \).

\[
S = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}
\]

(b) Based on the Jordan canonical form (JCF) obtained in (a), the minimal polynomial of \( A \) must be \( m_A(x) = (x - 3)^2 \).

(c) From the Division/Euclidean algorithm, \( f(z) = g(z)(z - 3)^2 + h(z) \), where \( h(z) \) is a degree 1 polynomial, say \( h(z) = az + b \) for some coefficients \( a, b \in \mathbb{C} \). Then

\[
f(A) = g(A)(A - 3I)^2 + h(A) = aA + bI = S(aJ - bI)S^{-1} = S \begin{pmatrix} 3a + b & a & 0 \\ 0 & 3a + b & 0 \\ 0 & 0 & 3a + b \end{pmatrix} S^{-1}
\]

If \( a = 0 \), then the Jordan canonical form of \( f(A) \) is \( \text{diag}(b, b, b) \). If \( a \neq 0 \),

\[
f(A) = S \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 1 \end{pmatrix} S^{-1} = \begin{pmatrix} 3a + b & 1 & 0 \\ 0 & 3a + b & 0 \\ 0 & 0 & 3a + b \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} S^{-1}
\]
2. We can determine the unique Jordan forms to permutation of the blocks. That is, there is no need to display all different permutations of the Jordan blocks.

Case 1: If the characteristic polynomial of \( A \) is \( \text{det}(zI - A) = (z - 1)^4(z - i) \).

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Case 2: If the characteristic polynomial of \( A \) is \( \text{det}(zI - A) = (z - 1)^3(z - i)^2 \).

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Case 3: If the characteristic polynomial of \( A \) is \( \text{det}(zI - A) = (z - 1)^2(z - i)^3 \).

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Case 4: If the characteristic polynomial of \( A \) is \( \text{det}(zI - A) = (z - 1)(z - i)^4 \).

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

3. Suppose \( A = S(J_2(i) \oplus J_2(1) \oplus J_1(1))S^{-1} \). So, the Jordan form of \( f(A) \) depends on \( f(J_2(i)), f(J_2(1)) \).

For \( f(J_2(i)) = h(J_2(i)) \) there are two possibilities depending on \( f(z) = (z - i)^2q(z) + a_1z + b_0 \).

1. If \( a_1 \neq 0 \), then the Jordan form of \( f(J_2(i)) = J_2(f(i)) \).

2. If \( a_1 = 0 \), then \( f(J_2(i)) = [f(i)] \oplus [f(i)] \).

Similarly, for \( f(J_2(1)) \) there are also two possibilities depending on \( f(z) = (z - 1)^2p(z) + b_1z + b_0 \).

1. If \( b_1 \neq 0 \), then the Jordan form of \( f(J_2(1)) = J_2(f(1)) \).

2. If \( b_1 = 0 \), then \( f(J_2(1)) = [f(1)] \oplus [f(1)] \).

Consequently there are 4 possible forms for \( f(A) = h(A) \) with eigenvalues \( f(i), f(i), f(1), f(1), f(1) \).

So, we have four possibilities for the JCF of \( f(A) \).

- If \((z - i)^2 \) is not a factor of \( f(z) \) and \((z - 1)^2 \) is not a factor of \( f(z) \), then \( f(A) \) is similar to
  \[
  J_2(f(i)) \oplus J_2(f(1)) \oplus J_1(f(1))
  \]

- If \((z - i)^2 \) is not a factor of \( f(z) \) and \((z - 1)^2 \) is a factor of \( f(z) \), then \( f(A) \) is similar to
  \[
  J_2(f(i)) \oplus J_1(f(1)) \oplus J_1(f(1)) \oplus J_1(f(1))
  \]
• If \((z - 1)^2\) is a factor of \(f(z)\) and \((z - 1)^2\) is a not factor of \(f(z)\), then \(A\) is similar to

\[ J_1(f(i)) \oplus J_1(f(i)) \oplus J_2(f(1)) \oplus J_1(f(1)). \]

• If \((x - i)^2(x - 1)^2\) is a factor of \(f(z)\), then \(f(A) = 0\).

4. Suppose \(f(z)\) is a polynomial, and \(A \in M_n\).

(a) First, note that for any positive integer \(k\), we have

\[ A^k x = A^{k-1}(Ax) = A^{k-1}(\lambda x) = \lambda (A^{k-1}x) = \lambda^2 (A^{k-2}x) = \cdots = \lambda^k x \]

Suppose \(f(z) = a_k z^k + \cdots + a_1 z + a_0\). Then

\[ f(A) x = (a_k A^k + \cdots + a_1 A + a_0 I)x \]

\[ = a_k (A^k x) + \cdots + a_1 (Ax) + a_0 x \]

\[ = a_k (\lambda^k x) + \cdots + a_1 (\lambda x) + a_0 x \]

\[ = (a_k \lambda^k + \cdots + a_1 \lambda + a_0)x \]

\[ = f(\lambda) \]

(b) Consider \(A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\) and \(f(z) = z^2\). Note that \(x = e_2\) is an eigenvector of \(f(A) = 0_2\) but \(Ax = e_1 \neq \lambda x\).

Alternatively, let \(A = \text{diag}(1, -1)\), then \(A^2 = I\). Then every nonzero vector in \(\mathbb{C}^2\) is an eigenvector of \(A^2\), but it is not true for \(A\).

5. Suppose \(A\) is \(m \times n\) and \(B\) is \(n \times m\). Without loss of generality, assume \(m \geq n\). Define the \(m + n\) matrices

\[ P = \begin{pmatrix} AB & 0 \\ B & 0_n \end{pmatrix}, \quad Q = \begin{pmatrix} 0_m & 0 \\ B & BA \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}. \]

Note that \(S\) is invertible since its upper triangular with nonzero diagonal entries. Furthermore

\[ PS = \begin{pmatrix} AB & 0 \\ B & 0_n \end{pmatrix} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix} = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0_m & 0 \\ B & BA \end{pmatrix} = SQ \]

Thus, \(P\) and \(Q\) are similar. Then

\[ z^m \det(zI_n - BA) = \det(zI_{m+n} - Q) = \det(zI_{m+n} - P) = z^n \det(zI_m - AB) \]

Thus \(\det(zI_m - AB) = z^{m-n} \det(zI_n - BA)\). Thus, the nonzero eigenvalues of \(AB\) are exactly the nonzero eigenvalues of \(BA\).

6. Suppose \(f(z) = z^n + a_1 z^{n-1} + \cdots + a_n\). Let \(A_f\) below be the companion matrix of \(f\).

\[ A_f = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad zI - A_f = \begin{pmatrix} z + a_1 & a_2 & \cdots & a_n \\ -1 & z & \cdots & 0 \\ 0 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 \end{pmatrix} \]

3
(a) (Induction on the degree of \( f \)) Suppose \( f(z) = z + a_1 \). Then \( A_f = [-a_1] \) and hence \( \det(zI - A_f) = z + a_1 \). Let \( n > 1 \) and suppose that for any \( g(z) \) of degree less than \( n \), it holds that \( \det(zI - A_g) = g \). Now let \( f(z) = z^n + a_1z^{n-1} + \cdots + a_n \), with \( a_i \) given by 

\[
\dim \text{Nul}(A - \lambda_i) \]

for distinct eigenvalues \( \lambda_1, \ldots, \lambda_k \). Note that for each \( i = 1, \ldots, k \), the number of Jordan blocks in the JCF of \( A \) corresponding to \( \lambda_i \) is given by \( \dim(\text{Nul}(A - \lambda_i)) \).

The first \( n-1 \) columns of \( A - \lambda_i I \) are clearly linearly independent. Hence rank \( (A) \geq n-1 \) and so \( \dim(\text{Nul}(A)) \leq 1 \). Since \( \lambda_i \) is an eigenvalue, then \( \dim(\text{Nul}(A)) \geq 1 \). Therefore \( \dim(\text{Nul}(A)) = 1 \). This shows that for each \( i = 1, \ldots, k \), the JCF of \( A \) only has one Jordan block \( J_n(\lambda_i) \). Thus \( m_A(z) = (z - \lambda_1)^{a_1} \cdots (z - \lambda_k)^{a_k} = f(z) \).

7. (Extra Credits) Suppose \( A = J_m(\lambda) \) and \( x'(s) = Ax(s) \). That is

\[
\begin{align*}
x'_1(s) &= \lambda x_1(s) + x_2(s) \\
x'_2(s) &= \lambda x_2(s) + x_3(s) \\
& \vdots \\
x'_{m-1}(s) &= \lambda x_{m-1}(s) + x_m(s) \\
x'_m(s) &= \lambda x_m(s)
\end{align*}
\]

When \( k = m \), the solution to the differential equation \( x'_m(s) = \lambda x_m(s) \) is

\[ x_m(s) = q_m(s)e^{s\lambda}, \quad \text{where } q_m(s) = c_m0 = x_m(0) \quad (\text{constant function}). \]

Let \( k < m \). Suppose that for \( t = k + 1, \ldots, m \), it holds that \( x_t(s) = q_t(s)e^{s\lambda} \) for some degree \( m - t \) polynomial \( q_t(s) \). Now \( x'_k(s) = \lambda x_k(s) + x_{k+1}(s) \), so

\[
x'_k(s) = \lambda x_k(s) + q_{k+1}(s)e^{s\lambda} \quad \iff \quad e^{-s\lambda}x'_k(s) - \lambda e^{-s\lambda}x_k(s) = q_{k+1}(s)
\]

\[
\iff \quad \frac{d}{ds} \left( e^{-s\lambda}x_k(s) \right) = q_{k+1}(s)
\]

\[
\iff \quad e^{-s\lambda}x_k(s) = \int q_{k+1}(s) \, ds
\]

\[
\iff \quad x_k(s) = e^{s\lambda} \left( \int q_{k+1}(s) \, ds \right)
\]

4
Note that

\[
\int q_{k+1}(s) \, ds = \int (c_{(k+1)0} + c_{(k+1)1}s + \cdots + c_{m-k-1,m-k-1}s^{m-k-1}) \, ds
\]
\[
= c_{k0} + c_{(k+1)0}s + c_{(k+1)1}s^2 + \cdots + c_{m-k-1,m-k-1}s^{m-k} = q_k(s)
\]

Thus \(x_k(s) = q_k(s)e^{s\lambda}\). By principle of mathematical induction, we obtain the desired conclusion.