1. (8 points) Let \( \mathbf{d} = (6, 3, 1)^t \) and \( \mathbf{a} = (8, 2, 0)^t \)

(a) Solving \( a \in [0, 1] \) such that \( \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix} \begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \), we have \( a = 2/3 \) so that \( T_1 = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix} \oplus [1] \) satisfies \( T_1(8, 2, 0)^t = (6, 4, 0)^t \). Similarly, \( \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \) so that \( T_2 = [1] \oplus \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} (6, 4, 0)^t = (6, 3, 1)^t \). Hence, \( T_2T_1\mathbf{a} = \mathbf{b} \).

(b) Suppose \( U_1 = \begin{pmatrix} \sqrt{2/3} & -\sqrt{1/3} \\ -\sqrt{1/3} & \sqrt{2/3} \end{pmatrix} \oplus [1] \). Then \( U_1 \text{diag} (8, 2, 0)U_1^t = \begin{pmatrix} 6 & 2\sqrt{2} \\ 2\sqrt{2} & 4 \end{pmatrix} \oplus [0] \).

Let \( U_2 = [1] \oplus \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \). Then

\[
U_2U_1 \text{diag} (8, 2, 0)U_1^tU_2^t = U_2 \begin{pmatrix} 6 & -2\sqrt{2} & 0 \\ -2\sqrt{2} & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} U_2^* = \begin{pmatrix} 6 & \sqrt{6} & \sqrt{2} \\ \sqrt{6} & 3 & \sqrt{3} \\ \sqrt{2} & \sqrt{6} & 6 \end{pmatrix}.
\]

Note that \( U_1, U_2 \) are orthogonal and so is \( U = U_2U_1 \). So, for \( P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \), we get the matrix

\[
A = PU \text{diag}(8, 2, 0)U^tP^t = \begin{pmatrix} 1 & \sqrt{3} & \sqrt{2} \\ \sqrt{3} & 3 & \sqrt{6} \\ \sqrt{2} & \sqrt{6} & 6 \end{pmatrix}
\]

satisfying the desired conditions (i.e. real symmetric, has eigenvalues (8, 2, 0) and diagonal entries (1, 3, 6)).

2. (8 points) Let \( \mathbf{x} = (x_1, \ldots, x_n)^t, \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n \) with positive entries.

(a) Suppose \( \mathbf{x} \) is obtained from \( \mathbf{y} \) by changing two of its entries \( y_p > y_q \) to \( y_p - d, y_q + d \) with \( d \in (0, y_p - y_q) \). That is,

\[
x_j = \begin{cases} 
    y_j & \text{if } j \notin \{p, q\} \\
    y_p - d & \text{if } j = p \\
    y_q + d & \text{if } j = q
\end{cases}
\]

Note that \( d, y_p - y_q - d \geq 0 \). Then

\[
x_px_q = (y_p - d)(y_q + d) = y_p y_q + d(y_p - y_q - d) \geq y_p y_q.
\]

Hence \( \prod_{j=1}^{n} x_j = x_p x_q \prod_{j=1,j\neq p,q}^{n} x_j = x_p x_q \prod_{j=1,j\neq p,q}^{n} y_j \geq \prod_{j=1}^{n} y_j \).

(b) Suppose \( \mathbf{x} < \mathbf{y} \). Then there exists a sequence \( (\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_{k+1}) \) such that \( \mathbf{y}_0 = \mathbf{y}, \mathbf{y}_{k+1} = \mathbf{x} \) and \( \mathbf{y}_{r+1} = (y_1^{(r+1)}, \ldots, y_n^{(r+1)}) \) is obtained from \( \mathbf{y}_r = (y_1^{(r)}, \ldots, y_n^{(r)}) \) by changing two of its entries as described in problem (a). Thus for any \( r = 0, \ldots, k \), we have

\[
\prod_{j=1}^{n} y_j^{(r)} \leq \prod_{j=1}^{n} y_j^{(r+1)} \implies \prod_{j=1}^{n} y_j = \prod_{j=1}^{n} y_j^{(0)} \leq \prod_{j=1}^{n} y_j^{(k+1)} = \prod_{j=1}^{n} x_j
\]
3. (8 points) (a) Suppose \( A = (a_{ij}) \in M_n \) is positive semidefinite. Let \( \lambda = (\lambda_1(A), \ldots, \lambda_n(A))^t \) be the vector of eigenvalues of \( A \) and \( d = (a_{11}, \ldots, a_{nn})^t \) be the vector of composed of the diagonal entries of \( A \). By Theorem 3.1.3, \( d \prec \lambda \).

If \( A \) has zero eigenvalues, then \( \det(A) = 0 \). Note that all diagonal entries of \( A \) must be nonnegative since \( A \) is positive semidefinite. Trivially, \( 0 = \det(A) \leq \prod_{j=1}^n a_{jj} \).

Suppose \( A \) is positive definite (all eigenvalues are positive). Note that this implies that the diagonal entries of \( A \) are all positive (The existence of a zero diagonal entry will imply that the entire column/row of that entry is zero). Applying problem 2b, we get

\[
\prod_{j=1}^n \lambda_j(A) = \det(A) \leq \prod_{j=1}^n a_{jj}.
\]

(b) Suppose \( B \in M_n \) has columns \( b_1, \ldots, b_n \).

Note that for any \( X \), \( \det(\bar{X}) = \overline{\det(X)} \) because in the determinant expansion, all numbers are replaced by its conjugates. Thus \( \det(B^*) = \det(B)^\ast = \det(B) \) and hence

\[
\det(B^*B) = \det(B^*) \det(B) = |\det(B)|^2
\]

Note that the \( j \)th diagonal entry of \( B^*B \) is the the product of the \( j \)th row of \( B^* \) and \( j \)th column of \( B \), i.e., \( b_j^*b_j = ||b_j||^2 \). Since \( B^*B \) is positive semidefinite, we can apply Problem 3a to get

\[
\det(B^*B) = |\det(B)|^2 \leq \prod_{j=1}^n ||b_j||^2
\]

Taking the square root of both sides, we get \( |\det(B)| \leq \prod_{j=1}^n ||b_j|| \).

4. (6 points) Let \( A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n \) be positive semidefinite with \( A_{11} \in M_k \). Then \( A_{11}, A_{22} \) are also positive semidefinite and \( A_{21} = A_{12}^\ast \). Suppose \( A_{11} = U_1D_1U_1^\ast \) and \( A_{22} = U_2D_2U_2^\ast \), where \( U_1 \) and \( U_2 \) are unitary and \( D_1, D_2 \) are diagonal with nonnegative entries. Then

\[
\hat{A} = \begin{pmatrix} U_1^\ast & 0 \\ 0 & U_2^\ast \end{pmatrix} A \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} = \begin{pmatrix} D_1 & U_1^\ast A_{12}U_2 \\ U_2^\ast A_{12}^\ast U_1 & D_2 \end{pmatrix}
\]

\( \hat{A} \) is still positive semidefinite since it is unitarily similar to \( A \). Applying problem 3a to \( \hat{A} \), we get

\[
\det(\hat{A}) = \det(A) \leq \det(D_1) \det(D_2) = \det(A_{11}) \det(A_{22})
\]

5. (8 points) Let \( A = A^\ast \in M_n \).

(\( \Leftarrow \)) Suppose \( A \) has \( p \) positive and \( q \) negative eigenvalues. By the spectral decomposition, there exists a unitary \( U \) such that

\[
U^\ast AU = \text{diag} (a_1, \ldots, a_p, -a_{p+1}, \ldots, -a_{p+q}, 0, \ldots, 0),
\]

where \( a_1, \ldots, a_{p+q} > 0 \). For \( j = 1, \ldots, n \), define

\[
d_j = \begin{cases} \frac{1}{\sqrt{a_j}} & \text{if } 1 \leq j \leq p + q \\ 1 & \text{if } j > p + q \end{cases}
\]
and define \( S = U \text{diag}(d_1, \ldots, d_n) \), which is invertible since \( U \) is unitary and \( d_j > 0 \) for all \( j \). Then \( S^*AS = I_p \oplus -I_q \oplus 0_{n-p-q} \).

\((\Rightarrow)\) Suppose \( S^*AS = I_p \oplus -I_q \oplus 0_{n-p-q} \) for some invertible \( S \). Partition \( S = [S_1 \ S_2 \ S_3] \) such that \( S_1 \in M_{n,p}, S_2 \in M_{n,q} \) and \( S_3 \in M_{n,n-p-q} \). Then

\[
\begin{pmatrix}
S_1^*AS_1 & S_1^*AS_2 & S_1^*AS_3 \\
S_2^*AS_1 & S_2^*AS_2 & S_2^*AS_3 \\
S_3^*AS_1 & S_3^*AS_2 & S_3^*AS_3
\end{pmatrix} = \begin{pmatrix}
I_p & 0 & 0 \\
0 & -I_q & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

By the QR decomposition theorem, there exists an invertible upper triangular \( R_1 \in M_p \) and an isometry \( V_1 \in M_{n,p} \), i.e. \( V_1^*V_1 = I_p \) such that \( S_1 = V_1R_1 \). Since \( S_1^* AS_1 = I_p \), then \( V_1^* AV = (R^*)^{-1} R^{-1} \), which is a positive definite \( p \times p \) matrix by Theorem 2.2.5 c. Now, using the min-max characterization of the eigenvalues of \( A \), we have

\[\lambda_p(A) \geq \lambda_p(V_1^*AV_1) = \lambda_p((R^*)^{-1} R^{-1}) > 0\]

Thus, \( A \) must have at least \( p \) positive eigenvalues.

Similarly, Let \( S_2 = V_2R_2 \) for some \( V_2 \in M_{n,q} \) satisfying \( V_2^*V_2 = I_q \) and an invertible upper triangular \( R_2 \in M_q \). Since \( S_2^*AS_2 = -I_q \), then \( V_2^*AV_2 = -(R_2^*)^{-1} R_2^{-1} \) which is negative definite (all eigenvalues are negative).

\[\lambda_{n-q+1}(A) \leq \lambda_1(V_2AV_2^*) = \lambda_1(-(R_2^*)^{-1} R_2^{-1}) < 0\]

Hence \( \lambda_{n-q+1}(A), \ldots, \lambda_n(A) \) — the last \( q \) eigenvalues of \( A \) are negative.

Now, since \( S \) is invertible, then rank \((A) = \text{rank}(S^*AS) = p + q \). Since \( A \) hermitian, this means that it is diagonalizable and its rank is equal to the number of its nonzero eigenvalues. Therefore \( A \) must have \( n-p-q \) zero eigenvalues. This forces the number of positive eigenvalues of \( A \) to be exactly \( p \) and the number of negative eigenvalues of \( A \) to be exactly \( q \).

6. (8 points) Lidskii’s inequality (Theorem 3.3.2) states that for any \( 1 \leq k \leq n \) and \( 1 \leq r_1 < r_2 < \cdots < r_k \leq n \), it holds that for any \( n \times n \) Hermitian matrices \( X \) and \( Y \)

\[
\sum_{j=1}^{k} \lambda_{r_j}(X + Y) \leq \sum_{j=1}^{k} \lambda_{r_j}(X) + \lambda_j(Y)
\]

Suppose \( A, B \in M_n \) are Hermitian matrices with eigenvalues \( a_1 \geq \cdots \geq a_n \) and \( b_1 \geq \cdots \geq b_n \), respectively. Let

\[
d = (a_1 + b_n, a_2 + b_{n-1}, \ldots, a_n + b_1) \quad \text{and} \quad u = (a_1 + b_1, \ldots, a_n + b_n).
\]

\((First, \ we\ will\ show\ that\ \lambda(A + B) < u)\)

Obviously, \( \sum_{j=1}^{n} \lambda_j(A + B) = \text{tr}(A + B) = \sum_{j=1}^{n} (a_j + b_j) \). Now, suppose \( 1 \leq k < n \). If we apply Lidskii’s inequality to \( (r_1, \ldots, r_k) = (1, \ldots, k) \), we get

\[
\sum_{j=1}^{k} \lambda_j(A + B) \leq \sum_{j=1}^{k} a_j + b_j = \text{sum of } k \text{ largest entries of } u
\]
This shows \( \lambda(A + B) < u \).

(Next, we will show that \( d < \lambda(A + B) \))

Obviously, \( \sum_{j=1}^{n} \lambda_j(A + B) = \sum_{j=1}^{n} (a_j + b_j) = \sum_{j=1}^{n} (a_j + b_{n-j+1}) \). Now, suppose \( 1 \leq k < n \). Let \((s_1, \ldots, s_n)\) be the rearrangement of \((1, \ldots, n)\) such that

\[
a_{s_1} + b_{n-s_1+1} \geq a_{s_2} + b_{n-s_2+1} \geq \cdots \geq a_{s_n} + b_{n-s_n+1}.
\]

Let \( C = A + B \) so that \( A = (-B) + C \). If we apply Lidskii’s inequalities (Theorem 3.3.2) with \( \{r_1, \ldots, r_k\} = \{s_1, \ldots, s_k\} \)

\[
\sum_{j=1}^{k} \lambda_{r_j}((-B) + C) \leq \sum_{j=1}^{k} \lambda_{r_j}(-B) + \lambda_j(C)
\]

The left hand side of this inequality is

\[
\sum_{j=1}^{k} \lambda_{r_j}((-B) + C) = \sum_{j=1}^{k} \lambda_{r_j}(A) = \sum_{j=1}^{k} a_{r_j} = \sum_{j=1}^{k} a_{s_j}
\]

while the right hand side is

\[
\sum_{j=1}^{k} \lambda_{r_j}(-B) + \lambda_j(C) = \sum_{j=1}^{k} -b_{n-r_j+1} + \lambda_j(A + B) = \sum_{j=1}^{k} -b_{n-s_j+1} + \lambda_j(A + B).
\]

Thus, \( \sum_{j=1}^{k} a_{s_j} \leq \sum_{j=1}^{k} -b_{n-r_j+1} + \lambda_j(A + B) \), which implies

\[
\sum_{j=1}^{k} \lambda_j(A + B) \geq \sum_{j=1}^{k} a_{s_j} + b_{n-s_j+1} = \text{sum of } k \text{ largest entries of } d
\]

Therefore, \( d < \lambda(A + B) \).