Math 408 Advanced Linear Algebra       Homework 6       Your Name

Eight points for each question.

1. Suppose $n = 3$. List all the Horn’s sequences $(u_1, u_2), (v_1, v_2), (w_1, w_2)$ of length 2, and list all the Thompson standard sequences $(u_1, u_2), (v_1, v_2)$ and $(w_1, w_2) = (u_1 + v_1 - 1, u_2 + v_2 - 2)$. Hint: See p.23-24 in http://ciklixx.people.wm.edu/teaching/math408/note.pdf
You should see six sets of Horn’s sequences $(u_1, u_2), (v_1, v_2), (w_1, w_2)$, and one of them not Thompson standard sequences.

2. Let $A, B, C = A + B ∈ M_n$ be Hermitian with eigenvalues $a_1 ≥ ⋯ ≥ a_n, b_1 ≥ ⋯ ≥ b_n$ and $c_1 ≥ ⋯ ≥ c_n$, respectively. Show that if $C = (c_{ij})$ then

\[ \sum_{j=1}^{k} c_{jj} ≤ \sum_{j=1}^{k} (a_j + b_j); \]

the equality holds if and only if $A = A_{11} ⊕ A_{22}, B = B_{11} ⊕ B_{22}$ with $A_{11}, B_{11} ∈ M_k$ such that $A_{11}$ and $B_{11}$ have eigenvalues $a_1 ≥ ⋯ ≥ a_k, b_1 ≥ ⋯ ≥ b_k$, respectively.

3. (Weyl’s inequalities.) Suppose $A, B, C = A + B ∈ M_n$ are Hermitian matrices. Let $\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\}, \{z_1, \ldots, z_n\}$ be orthonormal sets such that $Ax_j = λ_j(A)x_j, By_j = λ_j(B)y_j, Cz_j = λ_j(C)z_j$. Suppose $i, j$ are positive integers such that $i + j - 1 ≤ n$.

(a) Show that there is a unit vector $v ∈ V_1 ∩ V_2 ∩ V_3$, where $V_1 = \text{span} \{x_i, \ldots, x_n\}, V_2 = \text{span} \{y_j, \ldots, y_n\}$ and $V_3 = \{z_1, \ldots, z_{i+j-1}\}$. Hint: Show that $V_2 ∩ V_3$ has dimension at least $(n - j + 1) + (i + j - 1) - n = i$ by considering the null space of the matrix $[y_j \cdots y_n \ z_1 \cdots z_{i+j-1}]$, and then show that $V_1 ∩ (V_2 ∩ V_3)$ is not the zero space.

(b) Show that the vector $v$ found in (a) satisfies

\[ λ_{i+j-1}(C) ≤ v^* Cv, v^* Av ≤ λ_i(A), v^* Bv ≤ λ_j(B), \]

and deduce that

\[ λ_{i+j-1}(C) ≤ λ_i(A) + λ_j(B). \]

4. Show that $ℓ_p(v) ≥ ℓ_q(v)$ for any vector $v ∈ R^n$ if $1 ≤ p ≤ q ≤ ∞$. Hint: Only need to prove the result for vector with nonnegative entries; show that $f(v) = \sum v_j^p ≥ 1$ if $g(v) = \sum v_j^q - 1 = 0$. Use Lagrange multipliers to the function $L(μ, v) = f(v) - μg(v)$ and conclude that all nonzero entries of $v$ have to be the same so that $v = γ(1, \ldots, 1, 0, \ldots, 0)$.

5. Show that if $x, y ∈ R^n$ are vectors with positive entries such that $x ≺ y$, then $ℓ_p(x) ≤ ℓ_p(y)$ for any $p ≥ 1$.

Hint: We need only do the special case when $x$ is obtained from $y$ by changing two entries $y_i > y_j$ to $y_i - d, y_j + d$ for $d ∈ (0, y_i - y_j)$.
6. Let \( u = (u_1, \ldots, u_n)^t \) and \( v = (v_1, \ldots, v_n)^t \) be such that \((|u_1|, \ldots, |u_n|) \prec_w (|v_1|, \ldots, |v_n|)\). Show that there is a nonnegative integer \( m \) and a nonnegative \( d \) such that

\[
(|u_1|, \ldots, |u_n|, d, \ldots, d) \prec (|v_1|, \ldots, |v_n|, 0, 0, \ldots, 0).
\]

Deduce from the result of the previous problem that \( \ell_p(u) \leq \ell_p(v) \) for any \( p \geq 1 \).

(Extra credits) Alternatively, show that there is \( \hat{v} = (\hat{v}_1, \ldots, \hat{v}_n) \) such that \( \sum_{j=1}^k \hat{v}_j = \max\{\sum_{j=1}^k |v_j|, \ell_1(u)\} \). Then prove that \((|u_1|, \ldots, |u_n|) \prec (\hat{v}_1, \ldots, \hat{v}_n) \) and

\[
\ell_p(u) \leq \ell_p(\hat{v}) \leq \ell_p(v).
\]

7. (Extra credits) Suppose \( c_1 \geq a_1 \geq c_2 \geq a_2 \geq \cdots \geq a_{n-1} \geq c_n \geq a_n \) are \( 2n \) real numbers. Show that there is a nonnegative real vector \( v \in \mathbb{R}^n \) such that \( D + vv^t \) has eigenvalues \( c_1 \geq \cdots \geq c_n \). Assume that \( c_n \geq a_n > 0 \). By interlacing inequalities, there is \( \tilde{C} = \begin{pmatrix} D & y \\ y^t & a \end{pmatrix} \). Show that \( C = D + vv^t \) has eigenvalues \( c_1 \geq \cdots \geq c_n \).

(Extra credit) Suppose \( C = A + iB \) so that \( A \) and \( B \) are Hermitian matrices. Suppose \( A \) has eigenvalues \( a_1, \ldots, a_n \), and \( B \) has eigenvalues \( b_1, \ldots, b_n \) such that \( a_1^2 \geq \cdots \geq a_n^2 \) and \( b_1^2 \geq \cdots \geq b_n^2 \). If \( C \) has singular values \( s_1 \geq \cdots \geq s_n \), show that

\[
(a_1^2 + b_1^2, \ldots, a_n^2 + b_n^2) \prec (s_1^2, \ldots, s_n^2)
\]

and

\[
(s_1^2 + s_n^2, \ldots, s_n^2 + s_1^2) / 2 \prec (a_1^2 + b_1^2, \ldots, a_n^2 + b_n^2).
\]

Hint: \( A^2 + B^2 = (CC^* + C^*C) / 2 \).

8. (Extra credit) Suppose \( A = \begin{pmatrix} \tilde{A} & * \\ 0 & * \end{pmatrix} \). Show that

\[
s_1(A) \geq s_1(\tilde{A}) \geq s_2(A) \geq s_2(\tilde{A}) \geq \cdots \geq s_{n-1}(\tilde{A}) \geq s_n(A).
\]

Hint: Apply interlacing inequalities to \( A^*A \).

9. (Extra credit) Suppose \( A = \begin{pmatrix} \tilde{A} & * \\ 0 & * \end{pmatrix} \). Show that

\[
s_1(A) \geq s_1(\tilde{A}) \geq s_2(A) \geq s_2(\tilde{A}) \geq \cdots \geq s_{n-1}(\tilde{A}) \geq s_n(A).
\]

Hint: Apply interlacing inequalities to \( A^*A \).

10. (Extra credit) Suppose \( A, B \in M_n \). For any subsequences \( (u_1, \ldots, u_k), (v_1, \ldots, v_k) \) and \( (w_1, \ldots, w_k) \) of \( (1, \ldots, n) \) such that \( w_j = u_j + v_j - j \) for \( j = 1, \ldots, k \), and \( u_k + v_k - k \leq n \), we have

\[
\prod_{j=1}^k s_{u_j}(A)s_{v_j}(B) \geq \prod_{j=1}^k s_{w_j}(AB).
\]

Hint: By induction on \( n \). Check the case for \( n = 2 \). Assume that the result holds for matrices of size \( n - 1 \). If \( k = n \), the equality holds. Suppose \( k < n \). Let \( p \) be the largest integer such that \( u_j = j \) for all \( j = 1, \ldots, p \), and \( q \) be the largest integer such that \( v_j = j \) for all \( j = 1, \ldots, q \). We may assume that \( q \leq p \). Let \( \tilde{C} = AB \), \( \{u_1, \ldots, u_n\} \) and \( \{v_1, \ldots, v_n\} \) be orthonormal sets such that

\[
B^*Bu_j = s_j(B)^2u_j \quad \text{and} \quad C^*Cv_j = s_j(C)^2v_j.
\]
Suppose $U, V$ are unitary such that the first $n - 1$ columns span a subspace containing $v_1, \ldots, v_1, u_{q+2}, \ldots, u_n$, and $V^*BU = \left( \begin{array}{cc} \tilde{B} & * \\ 0 & * \end{array} \right)$ with $\tilde{B} \in M_{n-1}$. Let $W$ be unitary such that $W^*BV = \left( \begin{array}{c} \tilde{A} \\ 0 \end{array} \right)$. Then $W^*ABV = \left( \begin{array}{c} \tilde{A} \tilde{B} \\ 0 \end{array} \right)$. Apply induction assumption on $\tilde{A} \tilde{B}$ to finish the proof.