1. Suppose $\| \cdot \|$ is a matrix/algebra norm on $M_n$, and suppose $A \in M_n$.
   (a) Let $x$ be an eigenvector of an eigenvalue $\mu$ of $A$, and let $X \in M_n$ be such that every column of $X$ equals $x$. Show that $AX = \mu X$.
   (b) Deduce from (a) that $r(A) \leq \|A\|$.

2. A norm $\nu$ on $F^n$ is compatible with a norm $\| \cdot \|$ on $M_n$ if $\nu(Ax) \leq \|A\| \nu(x)$ for all $x \in F^n$. Suppose $\| \cdot \|$ is an algebra norm on $M_n$. Show that the norm $\nu(x) = \| \begin{bmatrix} x & \cdots & x \end{bmatrix} \|$ on $F^n$ is compatible with $\| \cdot \|$.

3. Show that the following conditions are equivalent for any unitarily invariant norm $\| \cdot \|$ on $M_n$.
   (a) $\| \cdot \|$ is an algebra norm.
   (b) $\|E_{11}\| \geq 1$.
   (c) $\|A\| \geq s_1(A)$ for any $A \in M_n$.
   [Hint: Note that for any $A, B \in M_n$, $\|A\| \geq \|s_1(A)E_{11}\|$, and $\|AB\| \leq \|s_1(A)B\|$.
   
   Remark From the above result, one sees that for any unitarily invariant norm $\| \cdot \|$ on $M_n$, $\xi \| \cdot \|$ is an algebra UI norm if and only if $\xi \geq 1/\|E_{11}\|$.

4. (a) Let $n \geq m \geq 0$. Show that there is $A \in M_n$ such that $\lim_{k \to \infty} A^k$ exists and has rank $m$.
   (b) Give an example of $B \in M_n$ such that $\lim_{k \to \infty} s_1(B^k) = 1$ but $\lim_{k \to \infty} B^k$ does not exist.

5. Suppose $\|x\|$ is a norm on a complex linear space $V$ satisfying the parallelogram identity
   \[ \|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \]
   for all $x, y \in V$.
   Define $\langle x, y \rangle = a + ib$ with $2a = \|x + y\|^2 - \|x\|^2 - \|y\|^2$, $2b = \|x + iy\|^2 - \|x\|^2 - \|y\|^2$ for any $x, y \in V$.
   (a) Show that $\langle x, y \rangle$ is indeed an inner product.
   (b) Show that $\|x\| = \langle x, x \rangle^{1/2}$.

6. Suppose $A \in M_n$.
   (a) Show that there is a rank one matrix $T$ with $s_1(T) = s_n(A)$ such that $A - T$ is singular.
   (b) Show that if $R \in M_n$ satisfies $s_n(R) > s_1(A)$, then $A - R$ is invertible.

7. Suppose $A \in M_n$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ with real parts $a_1 \geq \cdots \geq a_n$. Show that for any $\xi > a_1$ then there is a Hermitian matrix $B \in M_n$ with $\lambda_1(B) < \xi$ such that $A - B$ has all eigenvalues lying in $\{ \mu \in \mathbb{C} : \mu + \bar{\mu} < 0 \}$.

8. (extra credits) (a) Show that the numerical radius is not an algebra norm.
   (b) Show that the norm defined by $\|A\| = \xi w(A)$ is an algebra norm if and only if $\xi \geq 4$. 
9. (extra credits) Suppose \( c = (c_1, \ldots, c_n)^t \) with \( c_1 \geq \cdots \geq c_n \geq 0 \), and \( x \in \mathbb{C}^n \). Show that

\[
\nu_c(x) = \max\{|e^tPx| : P \in GP_n\} = \sum_{j=1}^n c_j Qx
\]

for some \( Q \in GP_n \) such that \( Qx \) has nonnegative entries arranged in descending order.

Recall that \( GP_n \) is the set of matrices that can be written as the product of a permutation matrix and a diagonal unitary matrix. Hint: Let \( Q \in GP_n \) such that \( \nu_c(x) = |e^tQx| \). Show that we may assume \( Qx \) has nonnegative entries. Then show \( Qx = (x_1, \ldots, x_n)^t \) satisfies \( x_i \geq x_{i+1} \) for any \( i = 1, \ldots, n-1 \).

10. (extra credits) Recall that for any \( A \in M_n \), \( W(aI + bA) = \{a + b\mu : \mu \in W(A)\} = a + bW(A). \) Also, \( W(A) = W(U^*AU) \) for any unitary \( U \in M_n \). Thus, for any \( A \in M_2 \), we may replace \( A \) by \( A_0 = U^*(aA + bI)U \) and assume that \( A_0 = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \) with \( a, b \geq 0 \). It is known that if \( b = 0 \) then \( W(A_0) \) is a line segment joining \( a, -a \); if \( a = 0 \), then \( W(A_0) \) is a circular disk centered at \( 0 \) with radius \( |b|/2 \). Suppose \( ab \neq 0 \).

(a) Show that \( A_0 \) is unitarily similar to \( H + iG \) with \( H = \text{diag}(\alpha, -\alpha) \) and \( G = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \), where \( \alpha = \sqrt{a^2 + b^2}/4 \) and \( \beta = b/2 \).

(b) Show that

\[
W(A_0) = e^{it} \{v^*Hv + iv^*Gv : v \in \mathbb{C}^n, v^*v = 1\} = e^{it} \{ax + i\beta y : x + iy \in W(B_0)\},
\]

where \( B_0 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \) has numerical range equal to \( \{\mu : |\mu| \leq 1\} \).

(c) Deduce from (b) that

\[
W(A_0) = e^{it} \{x + iy : (x/\alpha)^2 + (y/\beta)^2 \leq 1\}
\]

is an elliptical disk with foci \( \pm \sqrt{\alpha^2 - \beta^2} = \pm a \), and semi-minor axis of length \( \beta = b/2 \).

**Remark** The above result is known as the elliptical range theorem asserting that the numerical range of \( A \in M_2 \) is always an elliptical disk with its eigenvalues \( \mu_1, \mu_2 \) as foci and length of minor axis equal to \( \sqrt{(\text{tr} AA^*) - |\mu_1|^2 - |\mu_2|^2} \).

11. (extra credits) Suppose \( (x, y) \) is an inner product on \( \mathbb{C}^n \). Let \( P = (p_{ij}) \in M_n \) be such that \( p_{ij} = \langle e_j, e_i \rangle \) for \( 1 \leq i, j \leq n \), where \( \{e_1, \ldots, e_n\} \) is the standard basis for \( \mathbb{C}^n \). Show that \( (x, y) = y^*Px \) and \( P \) is a positive definite matrix.

Conversely, suppose \( Q \in M_n \) is a positive definite matrix. Show that \( (x, y) = y^*Qx \) is an inner product on \( \mathbb{C}^n \).