1. Let $A = (a_{ij}) \in M_n$ be such that $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$ for every $i = 1, \ldots, n$.

(a) For each $i \in \{1, \ldots, n\}$, define $G_i(A) = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}$. Now, if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for every $i = 1, \ldots, n$, then $0 \notin \bigcup_{i=1}^{n} G_i(A)$. By Theorem 5.1.1 (Gershgorin Theorem), all eigenvalues of $A$ are contained in $\bigcup_{i=1}^{n} G_i(A)$. Thus, 0 is not an eigenvalue of $A$. Therefore, $A$ must be invertible.

(b) Suppose $a_{ii} \neq 0$ for all $i$ and $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all but one $i$'s, without loss of generality say at $i = 1$ (otherwise, apply a permutation similarity). That is $|a_{11}| = \sum_{j \neq 1} |a_{1j}|$.

Note that for any $i \in \{2, \ldots, n\}$ such that $a_{i1} \neq 0$, we have

$$1 < \frac{|a_{ii}| - \sum_{j \neq 1, i} |a_{ij}|}{|a_{i1}|}.$$ 

Thus, there must be a $\varepsilon > 1$ such that for all such $i$,

$$\frac{1}{a_{i1}} \sum_{j \neq 1} |a_{1j}| = 1 < \varepsilon < \frac{\varepsilon^n - \sum_{j \neq 1, i} |a_{ij}|}{|a_{i1}|}.$$ 

Then define $B = (b_{ij}) = D^{-1}AD$, where $D = \text{diag}(\varepsilon, 1, \ldots, 1)$. Then

$$\sum_{j \neq i} |b_{ij}| = \frac{1}{\varepsilon} |a_{i1}| < |a_{ii}| = b_{ii}$$

and for all $i > 1$, we have

$$\sum_{j \neq i} |b_{ij}| = \left( \sum_{j \neq 1, i} |a_{ij}| \right) + \varepsilon a_{i1} < a_{ii} = b_{ii}$$

(c) (Extra 4 points) The conclusion (b) fails if we do not assume that $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$ for every $i = 1, \ldots, n$. A counterexample is $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Note that $|a_{11}| < |a_{12}|$ and $|a_{22}| > |a_{21}|$ but $A$ is not invertible.

2. Let $A = (a_{ij}) \in M_n$ with an eigenvalue $\lambda$ and $x$ be an eigenvector of $A$ corresponding to $\lambda$. Suppose $D = \text{diag}(a_{11}, \ldots, a_{nn})$ and $B = A - D$. Then $Bx = Ax - Dx = \lambda x - Dx = (\lambda I - D)x$.

If $\lambda \neq a_{ii}$ for $i = 1, \ldots, a_{nn}$, then the diagonal matrix $\lambda I - D$ is invertible and hence

$$(\lambda I - D)^{-1}Bx = (\lambda I - D)^{-1}(\lambda I - D)x = Ix = x.$$ 

Therefore $x$ is an eigenvector of $(\lambda I - D)^{-1}B$ corresponding to the eigenvalue 1.
3. Let $A \in M_n$ with columns $v_1, \ldots, v_n$. Let $B = A^*A = (b_{ij})$ and note that

$$\det(B) = \det(A^*A) = \overline{\det(A)} \det(A) = |\det(A)|^2.$$ 

Since $B$ is positive semidefinite (by Theorem 2.2.5), we can apply HW 5 Problem 3 to conclude that

$$|\det(A)|^2 = \det(B) = \prod_{j=1}^n b_{jj} = \prod_{j=1}^n \sum_{k=1}^n a_{kj}a_{jk} = \prod_{j=1}^n \ell_2(v_j)$$

Taking square roots we get, $|\det(A)| \leq \prod_{j=1}^n \ell_2(v_j)$. By HW 7 Problem 4, we have $\ell_2(v_j) \leq \ell_1(v_j)$. Thus $|\det(A)| \leq \prod_{j=1}^n \ell_1(v_j)$. Applying the above result to $A^t = [y_1 \mid \cdots \mid y_n]$, we get

$$|\det(A)| = |\det(A^t)| \leq \prod_{j=1}^n \ell_1(y_j) = \prod_{j=1}^n \sum_{k=1}^n |a_{jk}|$$

4. Consider $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$. Suppose $\mu$ is a zero of $f(z)$.

(a) Note that $\mu$ is an eigenvalue of the companion matrix of $f$, that is

$$C_f = \begin{bmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_n \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} = (c_{ij})$$

Define the Gershgorin discs $G_j$ centered at the $j^{th}$ diagonal entry of $C_f$ with radius equal to absolute row sum of the $j^{th}$ row. By the Gershgorin theorem, $\mu \in \bigcup_{j=1}^n G_j$.

If $\mu \in G_i$ for $i > 1$, then $|\mu - c_{ii}| = |\mu| \leq \sum_{j \neq i} |c_{ij}| = 1$. On the other hand, if $\mu \in G_1$,

then $|\mu - c_{11}| = |\mu + a_1| \leq \sum_{j \neq 1} |c_{1j}| = \sum_{k=2}^n |a_k|$. Using triangle inequality

$$|\mu| = \leq |\mu + a_1| + |a_1| \leq \sum_{k=1}^n |a_k|.$$ 

Thus, $|\mu| \leq \max\{1, \sum_{j=1}^n |a_j|\}$.

We can use the same arguments above for $C_f^t$, which has the same eigenvalues as $C_f$. Define the Gershgorin discs $\tilde{G}_j$ centered at the $j^{th}$ diagonal entry of $C_f$ with radius equal to absolute column sum of the $j^{th}$ column. By the Gershgorin theorem, $\mu \in \bigcup_{j=1}^n \tilde{G}_j$. If $\mu \in \tilde{G}_1$, then $|\mu| \leq |\mu + a_1| + |a_1| \leq 1 + |a_1|$. If $\mu \in \tilde{G}_n$, then $|\mu| \leq |a_n|$. Lastly, if $\mu \in G_j$ for $1 < j < n$, then $|\mu| \leq 1 + |a_j|$. Therefore, $|\mu| \leq \max(|a_j| : 1 \leq j \leq n-1) \cup \{|a_n|\}$. 

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(b) Let \( v^* = [-a_1 \ldots -a_{n-1}] \) so that \( C_f = \begin{bmatrix} v^* & -a_n \\ I_{n-1} & 0 \end{bmatrix} \). Then \( C_fC_f^* = \begin{bmatrix} v^*v + |a_n|^2 & v^* \\ v & I_{n-1} \end{bmatrix} \).

Note that \( I_{n-1} \) is a principal submatrix of \( C_fC_f^* \). Using Theorem 3.4.2 (interlacing theorem), we get

\[
s_1^2(C_f) \geq \lambda_1(I_{n-1}) \geq s_2^2(C_f) \geq \lambda_2(I_{n-1}) \geq \cdots \geq s_{n-1}^2(C_f) \geq \lambda_{n-1}(I_{n-1}) \geq s_n^2(C_f)
\]

\[
= 1 \text{ for } j = 1 \text{ or } 2.
\]

Therefore, \( s_2(C_f) = \cdots = s_{n-1}(C_f) = 1 \).

If \( \gamma = ||v|| = \sqrt{\sum_{j=1}^{n-1} |a_j|^2} = 0 \). Then it is obvious from the form of \( C_fC_f^* \) that \( \{ |a_n|, 1 \} = \{ s_1(C_f), s_n(C_f) \} \). If \( ||v|| \neq 0 \), let \( U \) be an \((n − 1) \times (n − 1)\) unitary matrix whose last column is \( \frac{v}{||v||} \). Consider

\[
B = \begin{bmatrix} 0 & I_{n-1} \\ 1 & 0 \end{bmatrix} C_f \begin{bmatrix} U & 0 \\ 0 & sgn(-a_n) \end{bmatrix} = \begin{bmatrix} I_{n-2} & 0 \\ 0 & X \end{bmatrix}, \text{ where } X = \begin{pmatrix} 1 & 0 \\ \gamma & |a_n| \end{pmatrix}.
\]

\( U \) is unitarily equivalent to \( C_f \), so \( B \) has the same singular values as \( C_f \). Therefore \( s_1(C_f) \) and \( s_n(C_f) \) are the singular values of \( X \).

(c) From (b), we can compute the singular values of \( X \) by looking at \( XX^* = \begin{bmatrix} 1 & \gamma \\ \gamma & \gamma^2 + |a_n|^2 \end{bmatrix} \)

which has characteristic polynomial \( p(x) = x^2 - (1 + \gamma^2 + |a_n|^2)x + |a_n|^2 \). And therefore

\[
s_1^2(C_f) = \frac{1 + \gamma^2 + |a_n|^2 + \sqrt{(1 + \gamma^2 + |a_n|^2)^2 - 4|a_n|^2}}{2}
\]

and

\[
s_n^2(C_f) = \frac{1 + \gamma^2 + |a_n|^2 + \sqrt{(1 + \gamma^2 + |a_n|^2)^2 - 4|a_n|^2}}{2}
\]

Next, consider

\[
\left( \frac{1}{2} \left( \sqrt{(1 + |a_n|^2)^2 + \gamma^2} + \sqrt{(1 - |a_n|^2)^2 + \gamma^2} \right) \right)^2
\]

\[
= \frac{1}{4} \left( 2 + 2|a_n|^2 + 2\gamma^2 + 2\sqrt{(1 + |a_n|^2)^2 + \gamma^2}(1 - |a_n|^2) + \gamma^2 \right)
\]

\[
= \frac{1}{4} \left( 1 + |a_n|^2 + \gamma^2 + \sqrt{(1 - |a_n|^2)^2 + \gamma^2} \right)^2
\]

\[
= \frac{1 + \gamma^2 + |a_n|^2 + \sqrt{(1 + |a_n|^2)^2 + \gamma^2}}{2} = s_1^2(C_f)
\]

Now, \( \mu \leq |\lambda_1(C_f)| \leq s_1(C_f) \). Thus, \( \mu \leq \frac{1}{2} \left( \sqrt{(1 + |a_n|^2)^2 + \gamma^2} + \sqrt{(1 - |a_n|^2)^2 + \gamma^2} \right) \).

5. Let \( A \in M_n \) with an eigenvalue \( \mu \) such that \( A - \mu I \) has rank \( n - 1 \). Then \( Nul(A - \mu I) \) and \( Nul(A^* - \mu I) \) both have dimension 1. Let \( x_1, y_1 \) be a right and left eigenvectors of \( A \) with \( y^*x_1 = 1 \), that is \( Nul(A - \mu I) = Span\{x_1\} \) \( Nul(A^* - \mu I) = Span\{y_1\} \). Suppose \( \{x_2, \ldots, x_n\} \) is a basis for \( Col(A - \mu I) = Nul(A^* - \mu I) \). So \( y_j^*x_j = 0 \) for \( j = 2, \ldots, n \). Then, \( \{x_1, \ldots, x_n\} \) must form a basis for \( \mathbb{C}^n \). Thus, the matrix \( S = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \) is invertible and \( S^{-1} \) has \( y_1^* \) as its first row. Denote the rows of \( S^{-1} \) by \( y_1^*, \ldots, y_n^* \). Then define \( B = S^{-1}AS = (y_i^*Ax_j) = (b_{ij}) \)

\[
b_{i1} = y_i^*Ax_1 = \mu y_i^*x_1 = \mu \delta_{i1}
\]

and

\[
b_{ij} = y_i^*Ax_j = \mu y_i^*x_j = \mu \delta_{ij}
\]

and thus \( B = [\mu] \oplus A_1 \), where \( rank(A - \mu I) = rank(B - \mu I) = rank([0] \oplus (A_1 - \mu I)) = n - 1 \). This implies \( A_1 - \mu I \) is invertible.
6. Let \( A \in M_n \) be a nonnegative matrix, then \( I + A \) is also nonnegative. Assume that \( (I + A)^k \) is positive. Then by Theorem 5.3.1, we conclude that \( I + A \) has a simple eigenvalue \( r(I + A) \) with positive right and left eigenvectors \( x \) and \( y \). However, since the eigenspace of \( I + A \) corresponding to an eigenvalue \( \lambda \) is the same as the eigenspace of \( A \) corresponding to the eigenvalue \( \lambda - 1 \). Thus, we conclude that \( A \) has a simple eigenvalue \( r(A) = r(A + I) - 1 \) with positive right and left eigenvectors \( x \) and \( y \).

**Proof:** Suppose \( A \in M_n \) is a nonnegative matrix is reducible. Then up to a permutation similarity, we have \( A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \) with \( A_{11} \in M_k \) with \( 1 \leq k \leq n - 1 \). Then \( (I_n + A)^{n-1} = \begin{pmatrix} (I_k + A_{11})^{n-1} & * \\ 0 & (I + A_{22})^{n-k} \end{pmatrix} \), which is not positive.

To prove the converse, we need to use some graph theory concept. Suppose \( A \) is irreducible. We can construct a graph \( G(I + A) \) with vertices \( 1, \ldots, n \) such that there is an arc \((i, j) \) from vertex \( i \) to vertex \( j \) if \( a_{ij} \neq 0 \) or \( i = j \). It is easy to check that there can go from vertex \( i \) to vertex \( j \) by forming a path \( i - j_1 - j_2 - j_3 - \cdots - j_k = j \) with \( k \) arcs \((i, j_1), (j_1, j_2), \ldots, (j_{k-1}, j_k) \) if and only if \((I + A)^k \) has a nonzero \((i, j) \) entry. Moreover, there is a path from vertex \( i \) to vertex \( j \) implies that there is a path from vertex \( i \) to vertex \( j \) in fewer than \( n - 1 \) steps.

Consider the set \( T_1 \) of the vertices \( j \) that can be reached from vertex \( 1 \) to \( j \) in \( k \) steps for \( k = 1, 2, \ldots \). We claim that \( T_1 = \{1, \ldots, n\} \). Otherwise, relabeling the vertices so that \( T_1 = \{1, \ldots, k\} \) with \( k < n \). Then permute the rows and columns of \( A \) accordingly. We will have \( A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \) with \( A_{11} \in M_k \). Note that \( A_{21} \) is nonzero. By permutation of the rows and columns with indices \( k + 1, \ldots, n \), we may assume that the first row of \( A_{21} \) is nonzero. Now, \( A^{k-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \) will be such that \( B_{11} \in M_k \) has positive entries so that \( A^k = AA^{k-1} \) has positive entries at the \((k + 1, 1)\) entry so that vertex \( k + 1 \) can be reached by \( 1 \) by a length \( k \) path, which is a contradiction. Thus, if \( A \) is irreducible, then \( T_1 = \{1, \ldots, n\} \), and \((I + A)^{n-1}\) has positive entries.

The first group

7. Let \( V \in M_{4,2} \) satisfy \( V^*V = I_2 \), and let \( P = VV^* \).

(a) \( V \) has a \( 2 \times 2 \) submatrix \( W \) with singular values \( s_1 \) and \( s_2 \) if and only if \( W \) can be written as \( W = SV \), where \( S = \begin{bmatrix} e_i^* \\ e_j^* \end{bmatrix} \) for some \( i, j \in \{1, \ldots, 4\} \) with \( i \neq j \) and \( WW^* \) has eigenvalues \( s_1^2 \) and \( s_2^2 \). Note that \( WW^* = SVV^*S^* \) is the principal submatrix of \( P \) indexed by \( i \) and \( j \).

(b) Note that \( P = VV^* \in M_4 \) and \( V^*V = I_2 \) have the same nonzero eigenvalues. Hence the eigenvalues of \( P \) must be 1 (with multiplicity 2) and 0 (with multiplicity 2). Thus, \( \det(\lambda I - P) = \lambda^2(\lambda - 1)^2 \). By the interlacing theorem we have that for any \( 2 \times 2 \) principal submatrix of \( P \),

\[
0 \leq s_1^2(P) \leq 1 \quad \text{and} \quad 0 \leq s_2^2(P) \leq 1
\]
Consider the compound matrix $C_2(P)$, which is a $6 \times 6$ matrix whose entries are the determinants of all possible $2 \times 2$ submatrices of $P$. Using 5.5.3, we get

$$C_2(P) = C_2(VV^*) = C_2(V)C_2(V^*) = xx^*$$

where $x \in \mathbb{C}^6$ and $x^*x$ is the sum of all the principal minors of $P$. By Theorem 5.5.2, $x^*x = 1$ = sum of $2 \times 2$ principal minors of $I_2$. Thus among, the largest principal minor must be greater than or equal to $\frac{1}{6}$. If this principal submatrix has eigenvalues $s_1^2$ and $s_2^2$, then $s_1^2 \leq 1$ and thus $\frac{1}{6} \leq s_1^2s_2^2 \leq s_2^2$.

(c) **(Extra credit open problem.)** The solution will earn you an A for the course.) Show that there is a $2 \times 2$ principal submatrix of $P$ with smaller eigenvalue larger than or equal to $1/4$.

(d) **(Extra credit open problem.)** The solution will earn you an A for the course, and a research paper.) Prove that for any $n \times k$ matrix $V$ such that $V^*V = I_k$. There is an $k \times k$ principal submatrix of $P = VV^*$ with smallest eigenvalue larger than or equal to $1/n$. 