Is in triangular with non-zero diagonal.

The system is always solvable.

---

Proof of Theorem 1.10

Let \( S = \begin{bmatrix} I_k & X \\ 0 & I_{n-k} \end{bmatrix} \) be such that

\[
\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I_k & X \\ 0 & I_{n-k} \end{bmatrix} = \begin{bmatrix} I_k & X \\ 0 & I_{n-k} \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}
\]

Therefore

\[
\begin{bmatrix} A_{11} & A_{11}X + A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & XA_{22} \\ 0 & A_{22} \end{bmatrix}
\]

So,\( \exists X \) such that \( A_{11}X + A_{12} = XA_{22} \) and \( A_{11}X - XA_{22} = -A_{12} \).

Such an \( X \) exists by lemma 1.9. So we get the desired \( S = \begin{bmatrix} I_k & X \\ 0 & I_{n-k} \end{bmatrix} \).
Lemma 1.9 Suppose $A \in M_m, B \in M_n$ have no common eigenvalues. Then for any $C \in M_{m,n}$ there is $X \in M_{m,n}$ such that $AX - XB = C$.

Proof. By the theory of linear equations.

Theorem 1.10 Suppose $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n$ such that $A_{11} \in M_k, A_{22} \in M_{n-k}$ have no common eigenvalue. Then $A$ is similar to $A_{11} \oplus A_{22}$.

Corollary 1.11 Suppose $A \in M_n$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then $A$ is similar to $A_{11} \oplus \cdots \oplus A_{kk}$ such that $A_{jj}$ has (only one distinct) eigenvalue $\lambda_j$ for $j = 1, \ldots, k$.

Proof of Lemma 1.9

May assume $A, B$ are in triangular form because of the following

Let $S_1 A S_1 = \hat{A}_{11}$ upper triangular

$S_2 \hat{A}_{11} S_2^{-1} = \hat{B}$ lower triangular

To solve $AX - XB = C$, i.e.

$S_1^{-1} (S_1 \hat{A}_{11} S_1^{-1} X - S_2 \hat{B} S_2) = S_1^{-1} C S_2$

$\hat{A}^{-1} S_1^{-1} X S_1 S_2^{-1} = S_1^{-1} C S_2 = C$

We only need to solve $\hat{A} Y + Y \hat{B} = C$, and then recover

Let $Y = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$, $\hat{A} = \begin{pmatrix} \hat{A}_{11} \\ \hat{A}_{21} \end{pmatrix}$, $\hat{B} = \begin{pmatrix} \hat{B}_{11} \\ \hat{B}_{21} \end{pmatrix}$

Then

$n \begin{pmatrix} \hat{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{pmatrix} = \begin{pmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{pmatrix} = n \begin{pmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{pmatrix}$.
Definition 1.12 Let $J_k(\lambda) \in M_k$ such that all the diagonal entries equal $\lambda$ and all super diagonal entries equal 1. Then $J_k(\lambda)$ is called an upper triangular Jordan block of $\lambda$ of size $k$.

Theorem 1.13 Every $A \in M_n$ is similar to a direct sum of Jordan blocks.

Proof. We may assume that $A = A_{11} \oplus \cdots \oplus A_{kk}$. Then we use a proof of Mark Wildon.

https://www.math.vt.edu/people/rcnardym/class_home/Jordan.pdf

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]
A SHORT PROOF OF THE EXISTENCE OF JORDAN NORMAL FORM

MARK WILDON

Let $V$ be a finite-dimensional complex vector space and let $T : V \to V$ be a linear map. A fundamental theorem in linear algebra asserts that there is a basis of $V$ in which $T$ is represented by a matrix in Jordan normal form

$$
\begin{pmatrix}
J_1 & 0 & \ldots & 0 \\
0 & J_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & J_k
\end{pmatrix}
$$

where each $J_i$ is a matrix of the form

$$
\begin{pmatrix}
\lambda & 1 & \ldots & 0 \\
0 & \lambda & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda
\end{pmatrix}
$$

for some $\lambda \in \mathbb{C}$.

We shall assume that the usual reduction to the case where some power of $T$ is the zero map has been made. (See [1, §58] for a characteristically clear account of this step.) After this reduction, it is sufficient to prove the following theorem.

\[ \mathcal{A} \subset \mathbb{C}^k \rightarrow \mathbb{C}^k \]

**Theorem 1.** If $T : V \to V$ is a linear transformation of a finite-dimensional vector space such that $T^m = 0$ for some $m \geq 1$, then there is a basis of $V$ of the form

$$
U_1, T U_1, \ldots, T^{a_1-1} U_1, \ldots, U_k, T U_k, \ldots, T^{a_k-1} U_k
$$

where $T^{a_i} U_i = 0$ for $1 \leq i \leq k$.

At this point all the proofs the author has seen (even Halmos' in [1, §57]) become unnecessarily long-winded. In this note we present a simple proof which leads to a straightforward algorithm for finding the required basis.

\[ S^{-1} A S = \begin{pmatrix}
J_{b_0} & 0 \\
0 & J_{c_0}
\end{pmatrix} \]

+ $\lambda I$

*Date: December 2007.*
Proof. We work by induction on \( \dim V \). For the inductive step we may assume that \( \dim V \geq 1 \). Clearly \( T(V) \) is properly contained in \( V \), since otherwise \( T^m(V) = \cdots = T(V) = V \), a contradiction. Moreover, if \( T \) is the zero map then the result is trivial. We may therefore assume that \( 0 \subset T(V) \subset V \). By applying the inductive hypothesis to the map induced by \( T \) on \( T(V) \) we may find \( u_1, \ldots, u_l \in T(V) \) so that

\[
  u_1, Tu_1, \ldots, T^{b_1-1}u_1, \ldots, u_l, Tu_l, \ldots, T^{b_l-1}u_l
\]

is a basis for \( T(V) \) and \( T^{b_i}u_i = 0 \) for \( 1 \leq i \leq l \).

For \( 1 \leq i \leq l \) choose \( u_i \in V \) such that \( Tu_i = v_i \). Clearly \( \ker T \) contains the linearly independent vectors \( T^{b_1-1}u_1, \ldots, T^{b_l-1}u_l \); extend these to a basis of \( \ker T \), by adjoining the vectors \( w_1, \ldots, w_m \), say. We claim that

\[
  u_1, Tu_1, \ldots, T^{b_1}u_1, \ldots, u_l, Tu_l, \ldots, T^{b_l}u_l, w_1, \ldots, w_m
\]

is a basis for \( V \). Linear independence may easily be checked by applying \( T \) to a given linear relation between the vectors. To show that they span \( V \), we use dimension counting. We know that \( \dim \ker T = l + m \) and that \( \dim T(V) = b_1 + \ldots + b_l \). Hence, by the rank-nullity theorem,

\[
  \dim V = (b_1 + 1) + \ldots + (b_l + 1) + m,
\]

which is the number of vectors in our claimed basis. We have therefore constructed a basis for \( V \) in which \( T \) is in Jordan normal form. \( \square \)

We end by remarking that this proof can be modified to avoid the preliminary reduction. Let \( \lambda \) be an eigenvalue of \( T \). By induction we may find a basis of \( (T - \lambda I)V \) in which the map induced by \( T \) on \( (T - \lambda I)V \) is in Jordan normal form. This basis can then be extended in a similar way to before to obtain a basis for \( V \) in which \( T \) is in Jordan normal form.

References


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