Lemma 2.11 Let $F \subseteq M_n$ be a commuting family. Then there is a unit vector $v \in \mathbb{C}^n$ such that $v$

**Proof:** Consider $V \subseteq \mathbb{C}^n$ s.t. $A(v) \in V \forall A \in F$. 

Want to show that $\exists v \in \mathbb{C}^n$ s.t. $Av = \lambda v$. 

Let $V$ be such a subspace of $\mathbb{C}^n$ with minimum positive dimension.

We will show that the dimension must be 1. 

Assume not. Then $A(v) \in V$ has dimension $k > 1$ for all $A \in F$.

Let $A_0 = S^{-1}AS = \begin{bmatrix} \frac{x}{\sqrt{k}} & 0 \\ 0 & \frac{y}{\sqrt{k}} \end{bmatrix}$ and let $Au = \lambda u$.

So that

\[
A_0 \begin{bmatrix} u \\ 0 \end{bmatrix} \in V.
\]

and

\[
A_0 S \begin{bmatrix} u \\ 0 \end{bmatrix} = S \begin{bmatrix} \frac{\lambda u}{\sqrt{k}} \\ 0 \end{bmatrix} = S \begin{bmatrix} \lambda u \\ 0 \end{bmatrix} \in V.
\]

Here I choose $A_0$ so that not all vectors in $V$ are eigenvectors of $A_0$.

Consider $W = \bigcap_{\forall A \in F} V$: $A_0 w = \lambda w$.

Then $A(w) \subseteq V \forall A \in F$.

$W$ has positive dimension $\Rightarrow S(w) \in W$.

and $W \subseteq V$.

Claim: for any $w \in W$, $A \in F$:

\[
A A_0 = A_0 A \Rightarrow A_0(Aw) = A A_0 w = A \lambda w = \lambda (Aw).
\]

So $A(w) \subseteq W \forall A \in F$. !!!! $W$ has

question. dimension 1!
Recall that $E_1, E_2, \ldots, E_m, E_n$ is the standard basis for $\mathbb{R}^m$ and $\mathbb{R}^n$.

### 2.4 Singular decomposition and polar decomposition

**Lemma 2.14** Let $A$ be a nonzero $m \times n$ matrix, and $u \in \mathbb{C}^m$, $v \in \mathbb{C}^n$ be unit vectors such that $|u^* A v|$ attains the maximum value. Suppose $U \in M_m$ and $V \in M_n$ are unitary matrices with $u$ and $v$ as the first columns, respectively. Then $U^* A V = \begin{pmatrix} u^* A v & 0 \\ 0 & A_1 \end{pmatrix}$.

**Proof**

Note that $|u^* A v| = M$. Let $U^* A V = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$. Consider $x = \begin{pmatrix} a_{11} \\ \vdots \\ a_{mn} \end{pmatrix}$ has non-zero entry $a_{ij} > 0$.

Consider $x_0 = \frac{x}{\|x\|}$ and let $\hat{u} = U x_0 = U \frac{x}{\|x\|}$. Suppose $x_0 \not= 0$ for $j > 1$.

Then we can let $V_0 = \begin{pmatrix} \overline{a_{11}} & \cdots & \overline{a_{1n}} \\ \vdots & \ddots & \vdots \\ \overline{a_{m1}} & \cdots & \overline{a_{mn}} \end{pmatrix}$ and $V = V V_0$.

Then

$$u^* A V = \sqrt{\sum_{i=1}^m a_{ij}^2} > |a_{11}| \Rightarrow \text{ causality.}$$
Theorem 2.15 Let $A$ be an $m \times n$ matrix. Then there are unitary matrices $U \in M_m, V \in M_n$ such that

$$U^*AV = D = \sum_{j=1}^{k} s_j E_{jj};$$

where

$$U = \begin{pmatrix} u_1 & \cdots & u_m \end{pmatrix}, \quad V = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix},$$

and the singular values of $A$

$$s_1 \geq \cdots \geq s_k > 0.$$

The singular values of $A$ are the square roots of the eigenvalues of $AA^*$ and $A^*A$.

$$V = [v_1 \ldots v_n]$$ is a basis for $C^n$.

$$U = [u_1 \ldots u_m]$$ is a basis for $C^m$.

$$T = U^*AV$$

$$T(X) = \theta X$$

$$[T]_{u_i,v_j} = \begin{pmatrix} s_i & 0 \\ 0 & 0 \end{pmatrix}$$

$$T(u_i) = s_i u_i, \quad \cdots \quad T(v_i) = s_k v_i.$$

$$T(v_j) = 0, \quad j = k+1, \ldots, n.$$

Proof of Theorem 2.15. Let $A \in M_{m,n}$.

Assume $A \neq 0$.

Then there are $u \in C^m, v \in C^n$ unit vectors such that

$$|u^*A v|$$ is max.

We can replace $v$ by $e^{i\theta}v$ so that

$$|u^*Av| = |u^*Av| \leq s_1.$$

Then for any $U \in M_m, V \in M_n$ with $u, v$ as first columns, we have

$$U^*AV = \begin{pmatrix} s_1 & 0 \\ 0 & U \end{pmatrix}.$$

If $A = 0$, we are done. If not, by induction/repeat the process to $A_1$ so that there are unitary $U_i \in M_{m-1}$, $V_i \in M_{n-1}$ such that

$$U_i^* A_i V_i = \begin{pmatrix} s_1 & U \\ 0 & 0 \end{pmatrix}.$$
Remark 2.16 Let \(U \) be formed by the first \(k\) columns \(u_1, \ldots, u_k\) of \(U\) and \(V\) be formed by the first \(k\) columns \(v_1, \ldots, v_k\) of \(V\). Then

\[
A = U \text{diag}(s_1, \ldots, s_k) V^T = \sum_{j=1}^{k} s_j u_j v_j^T
\]

Note that \(s_1^2, \ldots, s_k^2\) are the nonzero eigenvalues of \(AA^*\) and \(A^*A\).

Let \(\{v_1, \ldots, v_k\} \subseteq \mathbb{C}^n\) be an orthonormal set of eigenvectors corresponding to the nonzero eigenvalues \(s_1^2, \ldots, s_k^2\) of \(AA^*\). Let \(u_j = Av_j/s_j\). Then \(\{u_1, \ldots, u_k\} \subseteq \mathbb{C}^n\) is an orthonormal family such that \(A = \sum_{j=1}^{k} s_j u_j v_j^T\).

Similarly, let \(\{u_1, \ldots, u_k\} \subseteq \mathbb{C}^n\) be an orthonormal set of eigenvectors corresponding to the nonzero eigenvalues \(s_1^2, \ldots, s_k^2\) of \(AA^*\). Let \(v_j = A^*u_j/s_j\). Then \(\{v_1, \ldots, v_k\} \subseteq \mathbb{C}^n\) is an orthonormal family such that \(A = \sum_{j=1}^{k} s_j u_j v_j^T\).

Corollary 2.17 Let \(A \in M_n\). Then \(A = U^*P = QV\) such that \(U, V \in M_n\) are unitary, and \(P, Q\) are positive semidefinite matrices with eigenvalues equal to the singular values of \(A\).