Amidakuji, sorting algorithms, & permutations:

From recreational mathematics to research mathematics

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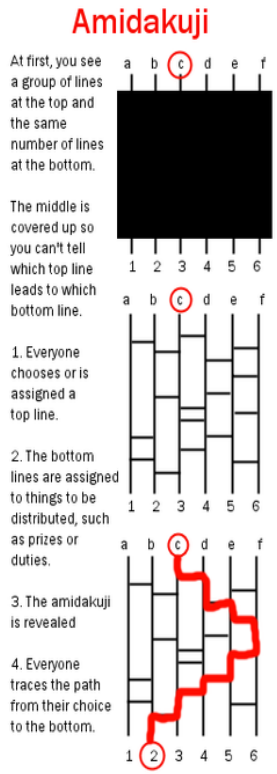
Amidakuji/Ghost Leg Drawing



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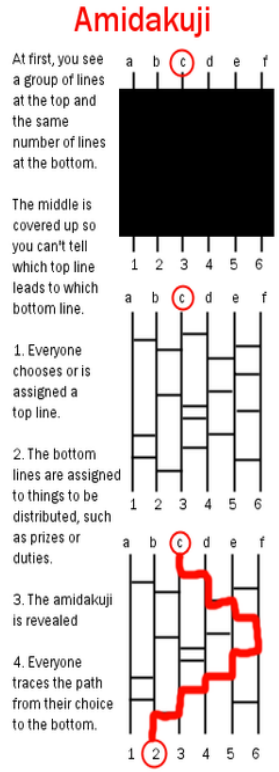

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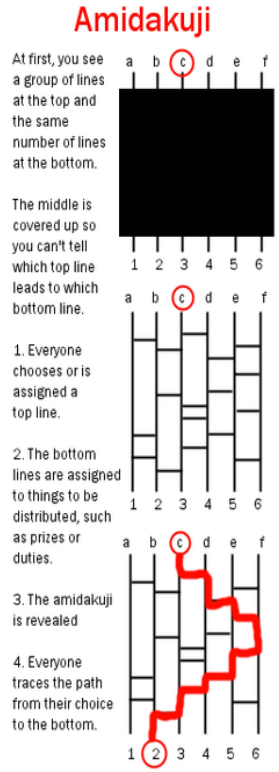

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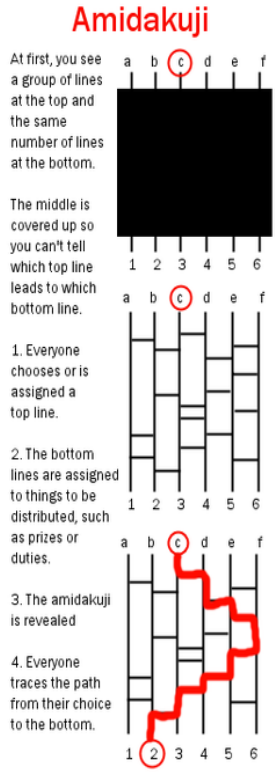

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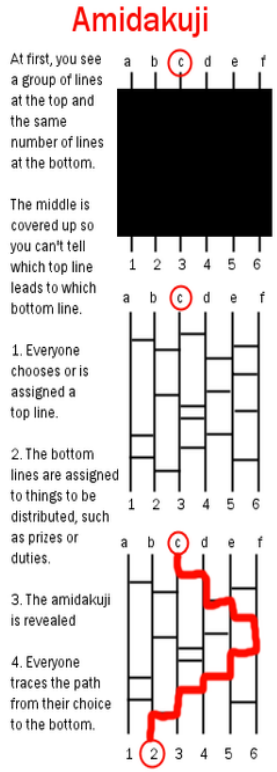

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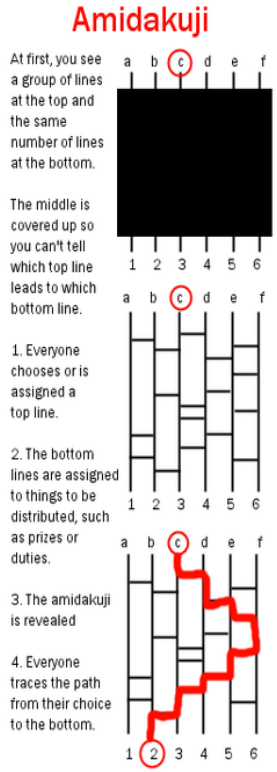

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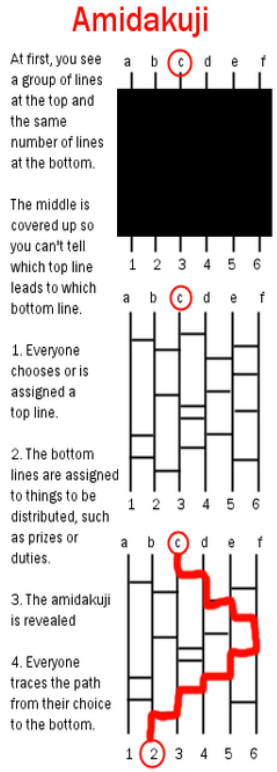

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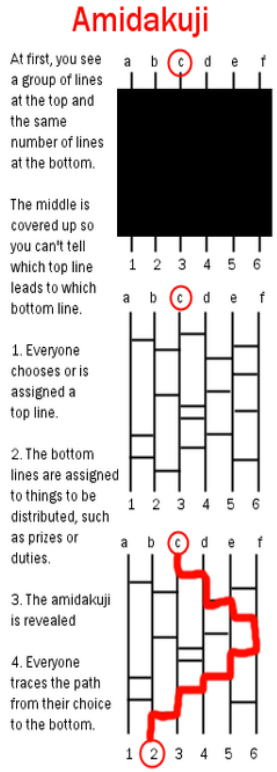

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- Why do we always get an one-one correspondence (bijection)?
- Can we get all possible job assignments?
- What is the minimum number of horizontal segments needed for a given job assignment?

Answer of Question 1

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- What if there is no horizontal line segment?
- What if there is one horizontal line segment?
- An easy induction argument!

Bubble sort

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$$\sigma = [i_1, \ldots, i_n] = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}.$$

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$$(n-1) + \cdots + 1 = n(n-1)/2 \text{ steps.}$$

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Example. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 9 & 6 & 7 & 1 & 8 & 2 \end{pmatrix} = (1, 3, 5, 6, 7)(2, 4, 9)(8).$

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Then $\sigma = (1, 7)(1, 6)(1, 5)(1, 3)(2, 9)(2, 4)$.

Some open problems

Let $1 \leq m < n$, and let G_m be the set of transpositions of the form $(i, i+\ell)$ with $1 \leq \ell \leq m$.

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- To find r^* and the permutation which is most difficult to get restore, we use the breadth first search.

Partial results of the general problem

We have the following list for $r^*(n,m)$ for S_n and $(i, i+\ell)$ with $\ell \le m$,

$n\backslash m$	1	2	3	4	5	6	7	8	9	10	11
2	1										
3	3	2									
4	6	4	3								
5	10	5	5	4							
6	15	[7]	6	6	5						
7	21	[10]	8	7	7	6					
8	28	[14]	[10]	9	8	8	7				
9	36	[16]	[11]	10	10	9	9	8			
10	45	[19]	[14]	[12]	11	11	10	10	9		
11	55	[23]	[16]	[14]	13	12	12	11	11	10	
12	66	29^*	20^*	17^*	16^*	14	13	13	12	12	11

where the entries marked by brackets are obtained by computer programming.

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- The Cayley digraph G generated by $T \subseteq S_n$:
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- If $T = \{(i, j) : 1 \leq i < j \leq n\}$, then $Diam(G) = n - 1$.

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- Open problem If $T = \{L, S\}$ with $L = (1, 2, \ldots, n)$ and $S = (1, 2)$, then the diameter is ???

S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}
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- Conjecture If $T = \{L, S, L^{-1}\}$, then the diameter is $\binom{n}{2}$ for $n \geq 4$, attained by the path joining $[2, 1, n, n-1, \ldots, 3]$ to the identity:

S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}
1	2	6	10	15	21	28	36	45	???

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Zejun Huang, Chi-Kwong Li, Sharon H. Li, Nung-Sing Sze, Factorization of permutations, preprint, arXiv:1303.3776.

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Thank you for your attention!