Hint for Problem 1.12

Note: normal A, i.e., $A^+A = AA^+$

$\Leftrightarrow$ there is unitary V s.t. $V^+AV = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$,

$\Leftrightarrow$ $A = V \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} V^+$.

Problem 1-12. $H = H^+$ so H is normal

$$\therefore H = V \begin{bmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{bmatrix} V^+$$

$$I \pm iH = VIV^+ \pm iV \begin{bmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{bmatrix} V^+$$

$$= V \begin{pmatrix} 1 \pm ih_1 & & 0 \\ & \ddots & \\ 0 & & 1 \pm ih_n \end{pmatrix} V^+$$

$$(I - iH)^{-1} = \left[V \begin{pmatrix} 1 - h_1 i & & 0 \\ & \ddots & \\ 0 & & 1 - h_n i \end{pmatrix} V^+ \right]^{-1}$$

$$= (V^+)^{-1} \begin{pmatrix} \frac{1}{1-h_1 i} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{1-h_n i} \end{pmatrix} V^{-1}$$

$$= V \begin{pmatrix} \frac{1}{1-h_1 i} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{1-h_n i} \end{pmatrix} V^+$$

$$(I + iH)(I - iH)^{-1}$$

$\cdots$

P

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

§1.6 & 1.8 Eigenvalues

Let A be an $n \times n$ complex matrix. Then $\lambda \in \mathbb{C}$ is an eigenvalue of A if there is a nonzero eigenvector $|\lambda\rangle$ such that $A|\lambda\rangle = \lambda|\lambda\rangle$.

- The matrix A always has complex eigenvalues because $\boxed{\det(tI - A) = 0}$ always has a solution λ so that $\boxed{(\lambda I - A)|\lambda\rangle = |0\rangle}$ has non-trivial solution.
- There is a unitary matrix U such that $U^\dagger AU = T$ is in upper triangular form. Moreover, $\det(tI - A) = \det(tI - T) = (t - T_{11}) \cdots (t - T_{nn})$.
- The matrix A is normal if and only if T is diagonal; the matrix A is unitary if and only if T is a diagonal matrix so that the diagonal entries have moduli 1; the matrix A is Hermitian if and only if T is a real diagonal matrices.

$\det(\lambda I - A) = \det(\lambda UU^\dagger - UTU^\dagger)$

$A = UTU^\dagger$

$= \det(U(\lambda I - T)U^\dagger)$

$= \det(U)\det(\lambda I - T)\det(U^\dagger)$

$= \det(\lambda I - T)$

Proof of the Schur Triangularization Lemma. Suppose $A \in M_n$. We show that there a unitary $U \in M_n$ such that $U^\dagger AU = T$ is in upper triangular form. We prove the result by induction on n. If $n = 1$, the result is trivial. Suppose the result holds for matrices in M_{n-1}. Let $A \in M_n$. Solve $\det(tI - A) = 0$ to get a solution λ.There is a nonzero vector $|\lambda\rangle$ such that $A|\lambda\rangle = \lambda|\lambda\rangle$. We may replace $|\lambda\rangle$ by $|\lambda\rangle/\||\lambda\rangle\|$ and assume that $|\lambda\rangle$ has unit length, and we can let $U_1 \in M_n$ be unitary with $|\lambda\rangle$ as the first column. Then $AU_1 = [A|x_1\rangle \cdots A|x_n\rangle] = [\lambda|x_1\rangle|A|x_2\rangle|\cdots|A|x_n\rangle]$ if U_1 has columns $|x_1\rangle = |\lambda\rangle, |x_2\rangle, \ldots, |x_n\rangle$. Because $\langle x_1|A|x_1\rangle = \lambda$, and $\langle x_j|A|x_1\rangle = 0$ for $j = 2, \ldots, n$, we see that $U^\dagger AU = \begin{pmatrix} \lambda_1 & * \\ 0 & A_1 \end{pmatrix}$. By induction assumption there is a unitary $U_2 \in M_{n-1}$ such that $U_2^\dagger A_1 U_2 = T_1$ in upper triangular form. Let $U = U_1 \begin{pmatrix} 1 & 0 \\ 0 & U_2 \end{pmatrix} \in M_n$. Then $U^\dagger U = I_n$ so that U is unitary, and $U^\dagger AU = \begin{pmatrix} 1 & 0 \\ 0 & U_2^\dagger \end{pmatrix} \begin{pmatrix} \lambda_1 & * \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & U_2 \end{pmatrix} = \begin{pmatrix} \lambda & * \\ 0 & T_1 \end{pmatrix}$ is upper triangular form. □

$\lambda_1 \langle x_j | \lambda_1 | x_1 \rangle$

$\det(\lambda I - T)$

$= \det \begin{pmatrix} \lambda - T_{11} & & * \\ & \ddots & \\ 0 & & \lambda - T_{nn} \end{pmatrix}$

$= (\lambda - T_{11}) \cdots (\lambda - T_{nn})$

Proof of the consequences for normal, unitary, Hermitian matrices. Let $U^\dagger AU = T$. Suppose A is normal, i.e., $A^\dagger A = AA^\dagger$. Then

$$T^\dagger T = (U^\dagger AU)^\dagger(U^\dagger AU) = (U^\dagger A^\dagger U)(U^\dagger AU) = U^\dagger A^\dagger AU = U^\dagger AA^\dagger U = TT^\dagger.$$

Now, the $(1,1)$ entry of $T^\dagger T$ is $|T_{11}|^2$ and the $(1,1)$ entry of $TT^\dagger$ is $|T_{11}|^2 + \cdots + |T_{1n}|^2$. So, $T_{12} = \cdots = T_{1n} = 0$. Now, consider the $(2,2), \ldots, (n,n)$ entries, we see that only the diagonal entries of T can be nonzero. So, T is a diagonal matrix. Conversely, if $U^\dagger AU = T$ is in diagonal form, then $A = UTU^\dagger$ so that $A^\dagger A = U^\dagger T^\dagger UU^\dagger TU = U^\dagger T^\dagger TU$ and $AA^t = U^\dagger TT^\dagger U$. Note that for a diagonal matrix T, we have $TT^\dagger = T^\dagger T$ is a diagonal matrix with diagonal entries $|T_{11}|^2, \ldots, |T_{nn}|^2$. So, $A^\dagger A = AA^\dagger$.

Now, if A is unitary, then A is normal and $T = U^\dagger AU$ is diagonal, and is unitary. So, $T^\dagger T = I_n$ implies $|T_{jj}|^2 = 1$ for all j. Conversely, if $A = U^\dagger TU$ such that T is diagonal with all diagonal entries satisfying $|T_{jj}| = 1$, then $A^\dagger A = UT^\dagger U^\dagger UTU^\dagger = UT^\dagger TU^\dagger = UU^\dagger = I_n$.

Finally, suppose A is Hermitian, then $A = A^\dagger$ so that the diagonal matrix T satisfies $T^\dagger = UA^\dagger U^\dagger = UAU^\dagger = T$ So, T is Hermitian and all the diagonal entries of T are real. Conversely, if T is a real diagonal matrix, then $T = T^\dagger$ so that $A^\dagger = U^\dagger T^\dagger U = U^\dagger TU = A$. □

$\begin{bmatrix} \bar T_{11} & 0 & 0 \\ \bar T_{12} & \bar T_{22} & 0 \\ \bar T_{13} & * & \bar T_{33} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & * \\ 0 & 0 & T_{33} \end{bmatrix} = \begin{pmatrix} |T_{11}|^2 & * \\ * & * \end{pmatrix}$

$\begin{pmatrix} T_{11} & T_{12} & * \\ 0 & T_{22} & * \\ 0 & 0 & T_{33} \end{pmatrix} \begin{pmatrix} \bar T_{11} & 0 & 0 \\ \bar T_{12} & \bar T_{22} & \\ \bar T_{13} & \bar T_{23} & \bar T_{33} \end{pmatrix} = \begin{pmatrix} |T_{11}|^2 + |T_{12}|^2 \end{pmatrix}$

§1.9 Spectral decomposition

$\rightarrow U\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}U^\dagger$

- Suppose A is normal and $A = UDU^\dagger$, where D has diagonal entries $\lambda_1, \ldots, \lambda_n$ and U has orthonormal columns $|\lambda_1\rangle, \ldots, |\lambda_n\rangle$. Then $A = \lambda_1|\lambda_1\rangle\langle\lambda_1| + \cdots + \lambda_n|\lambda_n\rangle\langle\lambda_n|$ so that

$$A^k = \sum_{j=1}^{n} \lambda_j^k |\lambda_j\rangle\langle\lambda_j| \qquad \text{and} \qquad e^A = \sum_{k=0}^{\infty} \frac{1}{k!}A^k = \sum_{j=1}^{n} e^{\lambda_j}|\lambda_j\rangle\langle\lambda_j|.$$

- Also, $A = \sum_\alpha \alpha P_\alpha$, where $P_\alpha = \sum_{\lambda_j=\alpha} |\lambda_j\rangle\langle\lambda_j|$. Then $A^k = \sum_\alpha \alpha^k P_\alpha$ and $e^A = \sum_\alpha e^\alpha P_\alpha$.

Proof. Because $A|\lambda_j\rangle = \lambda_j|\lambda_j\rangle$ for $j = 1, \ldots, n$, we have $AU = UD$ so that

$$A = UDU^\dagger = \sum_{j=1}^{n} \lambda_j|\lambda_j\rangle\langle\lambda_j|.$$

$A = \begin{bmatrix}|\lambda_1\rangle & |\lambda_2\rangle & \cdots & |\lambda_n\rangle\end{bmatrix}\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}\begin{bmatrix}\langle\lambda_1|\\ \vdots\\ \langle\lambda_n|\end{bmatrix}$

$= \begin{bmatrix}\lambda_1|\lambda_1\rangle & \cdots & \lambda_n|\lambda_n\rangle\end{bmatrix}\begin{bmatrix}\langle\lambda_1|\\ \vdots\\ \langle\lambda_n|\end{bmatrix}$

$= \lambda_1|\lambda_1\rangle\langle\lambda_1| + \cdots + \lambda_n|\lambda_n\rangle\langle\lambda_n|$

$a_0 I + a_1 A + \cdots + a_k A^k = (a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \cdots + a_k\lambda_1^k)|\lambda_1\rangle\langle\lambda_1| + \cdots +$

Now,

$$\boxed{A^k = \underbrace{UDU^\dagger \cdots UDU^\dagger}_{k} = UD^kU^\dagger = \sum_{j=1}^{n} \lambda_j^k|\lambda_j\rangle\langle\lambda_j|,}$$

and

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!}A^k = \sum_{k=0}^{\infty} \frac{1}{k!}\sum_{j=1}^{n} \lambda_j^k|\lambda_j\rangle\langle\lambda_j| = \sum_{j=1}^{n} e^{\lambda_j}|\lambda_j\rangle\langle\lambda_j|.$$

$\boxed{e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k}$

$\sum_{k=0}^{\infty} \frac{1}{k!}\lambda_1^k |\lambda_1\rangle\langle\lambda_1|$

$+ \sum \frac{1}{k!}\lambda_2^k |\lambda_2\rangle\langle\lambda_2|$

$+ \cdots$

$A = U\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}U^\dagger$

$A^k = U\begin{bmatrix}\lambda_1^k & & 0\\ & \ddots & \\ 0 & & \lambda_n^k\end{bmatrix}U^\dagger$

$= \begin{bmatrix}\lambda_1^k|\lambda_1\rangle & \lambda_2^k|\lambda_2\rangle & \cdots & \lambda_n^k|\lambda_n\rangle\end{bmatrix}\begin{bmatrix}\langle\lambda_1|\\ \vdots\\ \langle\lambda_n|\end{bmatrix}$

$\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}d_1 & 0\\ 0 & d_2\end{bmatrix}$

$= \begin{bmatrix}a_{11}d_1 & a_{12}d_2\\ a_{21}d_1 & a_{22}d_2\end{bmatrix}$

$\begin{bmatrix}r_1 & r_2\end{bmatrix}\begin{bmatrix}s_1\\ s_2\end{bmatrix}$

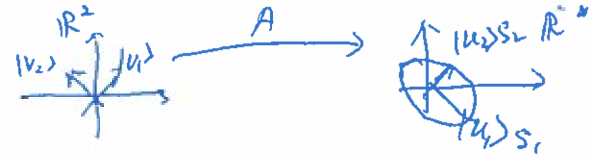
$= r_1 s_1 + r_2 s_2$

Remark: If $\lambda_1 = \lambda_2$

$\lambda_1|\lambda_1\rangle\langle\lambda_1| + \lambda_2|\lambda_2\rangle\langle\lambda_2|$

$= \lambda_1(|\lambda_1\rangle\langle\lambda_1| + |\lambda_2\rangle\langle\lambda_2|)$

$= \lambda_1 P_1$



$$AV = UD$$

$$A[|v_1\rangle \cdots |v_n\rangle] = [|u_1\rangle \cdots |u_n\rangle]$$

§1.9 Singular value decomposition (SVD)

Theorem *Every square matrix* A *can be written as* $A = UDV^\dagger$, *where* U, V *are unitary and* $D = \text{diag}\,(s_1, \ldots, s_n)$ *with* $s_1 \geq \cdots \geq s_n \geq 0$.

Proof. Proof by induction on the size of A. Clearly, the result holds for $n = 1$. Assume that the result is true for matrices of size less than n with $n \geq 2$. Choose unitary U, V, such that $S = U_1^\dagger A V_1$ has largest $(1,1)$ entry with largest real part S_{11}. First, we have $S_{11} \geq 0$, else replace V by $e^{it}V$ for some suitable e^{it}. Next, we claim that $S_{1j} = 0$ for $j = 2, \ldots, n$. If not, we can replace V_1 by $V_1 R$ where R is obtained from I_n by changing the 2×2 submatrix in rows and columns $1, j$ by $\frac{1}{\sqrt{S_{11}^2 + |S_{1j}|^2}} \begin{pmatrix} S_{11} & -S_{1j} \\ \bar{S}_{1j} & S_{11} \end{pmatrix}$. Then $U_1^\dagger A V_2 R$ has $(1,1)$ entry equal to $\sqrt{S_{11}^2 + |S_{1j}|^2} > S_{11}$, which is a contradiction. Similarly, we can show that $S_{j1} = 0$ for all $j = 2, \ldots, n$. Thus, $S = \begin{pmatrix} 1 & 0 \\ 0 & \hat{S} \end{pmatrix}$. By induction assumption, there are unitary U_2, V_2 such that $U_2^\dagger \hat{S} V_2 = \text{diag}\,(s_2, \ldots, s_n)$.

Let $V = V_1 \begin{pmatrix} 1 & \\ & V_2 \end{pmatrix}$ and $U = U_1 \begin{pmatrix} 1 & \\ & U_2 \end{pmatrix}$. Then $U^\dagger A V = \text{diag}\,(s_1, \ldots, s_n)$. $\square$

Note $s_1, \ldots, s_n$ are the singular values of A and equal the nonnegative square roots of the eigenvalues of $A^\dagger A$ or / and $AA^\dagger$.

The matrix A has rank k if and only if it has k nonzero singular values.

In practice, write $A^\dagger A = V D^2 V^\dagger$ so that the columns of the unitary matrix V, $|v_1\rangle, \ldots, |v_n\rangle$, are the right singular vectors. Then $AV = UD$ for some unitary U such that $A = UDV^\dagger$. In particular, if $s_j \neq 0$ then $|u_j\rangle = A|v_j\rangle / s_j$; if A has rank k so that $s_k > 0 = s_{k+1} = \cdots = s_n$ then $|u_{k+1}\rangle, \ldots, |u_n\rangle$ are chosen so that $\{|u_1\rangle, \ldots, |u_n\rangle\}$ is an orthonormal basis.

In Matlab, we use the command $[U, D, V] = \text{svd}(A)$.

$$S_{11} \cdot (-\bar{S}_{1j}) + (\bar{S}_{1j} \cdot S_{11})$$

$$A = UDV^\dagger$$

$$A^\dagger A = VDU^\dagger UDV^\dagger = VD^2V^\dagger$$

$$AA^\dagger = UD^2U^\dagger$$

$$\begin{bmatrix} s_1^2 & & 0 \\ & \ddots & \\ 0 & & s_n^2 \end{bmatrix}$$

$$\begin{pmatrix} S_{11} & \bar{S}_{1j} \end{pmatrix} \begin{pmatrix} -S_{1j} \\ S_{11} \end{pmatrix} = 0$$

$$U_1^\dagger A V_1 = \begin{pmatrix} S_{11} & S_{12} \\ & \end{pmatrix}$$

$$\begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{bmatrix} S_{11} \\ \bar{S}_{12} \end{bmatrix} \Big/ \sqrt{S_{11}^2 + |S_{12}|^2}$$

$m \times n$ B

§1.10 Tensor product (Kronecker product)

$m \times n$ $\quad p \times q$ $\quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$A = (A_{ij})$ $\quad B$ $\quad A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} \} 2m$

$A \otimes B : mp \times nq$ $\quad 2m$

- $A \otimes B = (A_{ij} B)$ satisfies
  (a) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$,
  (b) $A \otimes (B + C) = A \otimes B + A \otimes C$, $(A \otimes B)^{\dagger} = A^{\dagger} \otimes B^{\dagger}$, $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.
  (c) $\operatorname{tr}(A \otimes B) = (\operatorname{tr} A)(\operatorname{tr} B)$ and $\det(A \otimes B) = \det(A)^n \det(B)^m$ if $A \in M_m, B \in M_n$.

Proof. (a) Let A be $m \times n$, B be $r \times s$, C be $n \times p$, D be $s \times q$. Then $A \otimes B = (A_{ij} B)$ is $mr \times ns$, and $C \otimes D = (C_{ij} D)$ is $ns \times pq$. Now

$$(A \otimes B)(C \otimes D) = (A_{ij} B)(C_{ij} D) = (f_{rs} BD),$$

$\begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} \begin{bmatrix} C_{11}D & C_{12}D & C_{13}D \\ C_{21}D & C_{22}D & C_{23}D \end{bmatrix} = \big[(a_{11}C_{11} + a_{12}C_{21}) BD \quad BD$

where $f_{rs} = \sum_{\ell=1}^{n} A_{r\ell} C_{\ell s} = (AC)_{rs}$. So, $(f_{rs} BD) = AC \otimes BD$.

(b) $A \otimes (B + C) = (A_{ij}(B + C)) = (A_{ij} B) + (A_{ij} C) = A \otimes B + A \otimes C$.

$(A \otimes B)^{\dagger} = \overline{A \otimes B)^t = (A^t \otimes B^t)} = A^{\dagger} \otimes B^{\dagger}$.

$(A^{-1} \otimes B^{-1})(A \otimes B) = (A^{-1} A) \otimes (B^{-1} B) = I_m \otimes I_n = I_{mn} \implies A^{-1} \otimes B^{-1} = (A \otimes B)^{-1}$.

(c) Note that $\operatorname{tr}(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ji} = \operatorname{tr}(BA)$. Assume that A is $m \times m$ and B is $n \times n$. Let U, V be unitary and S, T be triangular such that $A = USU^{\dagger}$ and $B = VTV^{\dagger}$. Then $\operatorname{tr} A = \operatorname{tr} S, \operatorname{tr} B = \operatorname{tr} T$,

$$\operatorname{tr}(A \otimes B) = \operatorname{tr}((U \otimes V)(S \otimes T)(U \otimes V)^{\dagger}) = \operatorname{tr}(S \otimes T)$$

$$= \sum_{i=1}^{m} S_{ii} \sum_{j=1}^{n} T_{jj} (\sum_{i=1}^{m} S_{ii})(\sum_{j=1}^{n} T_{jj}) = (\operatorname{tr} S)(\operatorname{tr} T) = (\operatorname{tr} A)(\operatorname{tr} B).$$

Also,

$$\det(A \otimes B) = \det((U \otimes V)(S \otimes T)(U \otimes V)^{\dagger}) = \det(S \otimes T)$$

$$= \prod_{i=1}^{m} S_{ii}{}^{n} \det(T) = (\det S)^n (\det T)^m = (\det A)^n (\det B)^m.$$