$|x\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$ $\quad |\frac{1}{\sqrt2}|^2$ prob $\begin{pmatrix}1\\0\end{pmatrix}$ $\quad |\frac{1}{\sqrt2}|^2$ prob $\begin{pmatrix}0\\1\end{pmatrix}$

But $|x\rangle = \alpha_1|u_1\rangle + \alpha_2|u_2\rangle \neq \alpha_1|u_1\rangle + \alpha_2 e^{it}|u_2\rangle$

Remarks

1. The phase of the state does not matter, i.e., $|x\rangle$ and $e^{i\alpha}|x\rangle$ represents the same states.
2. In the finite dimensional case, if the state and the observable are represented by

$$|x\rangle = \sum_{j=1}^{n} c_j|u_j\rangle \in \mathbb{C}^n \quad \text{and} \quad A = \sum_{j=1}^{n} \lambda_j|u_j\rangle\langle u_j| = \sum_{j=1}^{n}\lambda_j P_j,$$

$= \sum_\alpha \alpha P_\alpha$

then the projective measurement of the state is

$$\langle x|A|x\rangle = \boxed{\sum_{j=1}^{n}\lambda_j|c_j|^2.}$$

Once the measurement is applied, the state becomes (collapses to)

$$\frac{P_i|x\rangle}{\sqrt{\langle x|P_i|x\rangle}} = \frac{P_i|x\rangle}{|c_i|}.$$

$\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \to x_1\begin{pmatrix}1\\0\end{pmatrix}$, $\frac{\quad}{|x_1|}$, $\quad x_2\begin{pmatrix}0\\1\end{pmatrix}$

3. In the Schrödinger equation, if $H(t)$ does not depend on t, then

$$|x(t)\rangle = e^{-iHt/\hbar}|x(0)\rangle.$$

Otherwise,

$$\boxed{|x(t)\rangle = \exp\left(\frac{-i}{\hbar}\int_0^t H(s)ds\right)|x(0)\rangle.}$$

Example If

$$H = \boxed{\frac{-\hbar}{2}w\sigma_x} \quad \text{and} \quad |\psi(0)\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{so that} \quad i\hbar\frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle,$$

then $|\psi(t)\rangle = \exp(i\frac{w}{2}\sigma_x t)|\psi(0)\rangle$. Hence, $\exp(-iHt/\hbar)$

$\begin{pmatrix}1\\0\end{pmatrix} \quad \frac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix} \quad \frac{1}{\sqrt2}\begin{pmatrix}1\\i\end{pmatrix}$

$$|\psi(t)\rangle = \underbrace{((\cos wt/2)I_2 + (i\sin wt/2)\sigma_x)}_{\begin{pmatrix}\cos wt/2 & -i\sin wt/2\\ i\sin wt/2 & \cos wt/2\end{pmatrix}}|\psi(0)\rangle = \begin{pmatrix}\cos wt/2\\ i\sin wt/2\end{pmatrix}.$$

$i\hbar\begin{pmatrix}\psi_1'(t)\\ \psi_2'(t)\end{pmatrix} = \frac{-\hbar}{2}\begin{pmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{pmatrix}\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix}$

$= -\frac{\hbar}{2}\begin{pmatrix}h_{11}\psi_1(t) + h_{12}\psi_2(t)\\ h_{21}\psi_1(t) + h_{22}\psi_2(t)\end{pmatrix}$

$x'(t) = kx(t)$

$x(t) = e^{+kt}x(0)$

$\begin{pmatrix}\psi_1(t)\\ \psi_2(0)\end{pmatrix}$

$\psi_1(t)$

$u^+\begin{pmatrix}\xi_1(t)\\ \xi_2(t)\end{pmatrix} = u^+\begin{pmatrix}e^{\frac{id_1}{2}} & 0\\ 0 & e^{\frac{id_2}{2}}\end{pmatrix}u\,u^+\begin{pmatrix}\xi_1(0)\\ \xi_2(0)\end{pmatrix}$

$u^+ i\hbar\begin{pmatrix}\psi_1'(t)\\ \psi_2'(t)\end{pmatrix} = \frac{-\hbar}{2}\begin{bmatrix}d_1 & \\ & d_2\end{bmatrix}u^+\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix}$

$\begin{pmatrix}\xi_1'(t)\\ \xi_2'(t)\end{pmatrix} = -\frac{\hbar}{2i}\begin{pmatrix}d_1 & 0\\ 0 & d_2\end{pmatrix}\begin{pmatrix}\xi_1(t)\\ \xi_2(t)\end{pmatrix}$

Two ways to compute

$$\exp(B) \quad \text{with} \quad B = (iwt/2)\sigma_x = (iwt/2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Method 1. Use the fact that

$$B = (iwt/2)\sigma_x = (iwt/2)P_1 + (-iwt/2)P_2 \quad \text{with } P_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, P_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Thus,

$$\exp(B) = e^{iwt/2}P_1 + e^{-iwt/2}P_2 = \frac{1}{2}\begin{pmatrix} e^{iwt/2} + e^{-iwt/2} & e^{iwt/2} - e^{-iwt/2} \\ e^{iwt/2} - e^{-iwt/2} & e^{iwt/2} + e^{-iwt/2} \end{pmatrix} = \begin{pmatrix} \cos(wt/2) & i\sin(wt/2) \\ i\sin(wt/2) & \cos(wt/2) \end{pmatrix}.$$

Method 2. Use the fact that $\sigma_x^n = I_2$ if n is even, and $\sigma_x^n = \sigma_x$ if n is odd. Then

$$\exp(B) = \sum_n B^n/n! = \sum_{2j} B^{2j}/(2j)! + \sum_{2j-1} B^{2j-1}/(2j-1)! = \cos(wt/2)I_2 + i\sin(wt/2)\sigma_x.$$

The uncertainty principle

Let $\mathrm{Exp}_x(A) = \langle x|A|x\rangle = \mu$ and

$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{2}{2} + \frac{1}{2} = \frac{3}{2}$

$$\mathrm{Var}_x(A) = \mathrm{Exp}_x((A - \mu I)^2) = \langle x|(A - \mu I)^2|x\rangle = \|(A - \mu I)|x\rangle\|^2.$$

In an deterministic model, the variance of measurements should go to zero as the apparatus is made very accurate.

Theorem For any observable A and B and for any state $|x\rangle$, we have

$$\boxed{\mathrm{Var}_x(A)\mathrm{Var}_x(B) \geq \frac{1}{4}\langle x|[A, B]|x\rangle,}$$

where $[A, B] = AB - BA$ is the commutator of A and B.

Bipartite systems dim dim

A system may have two components described by two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then the bipartite system is represented by $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. A general state in $\mathcal{H}$ has the form

$$|x\rangle = \sum_{i,j} c_{ij} |e_{1,i}\rangle \otimes |e_{2,j}\rangle \quad \text{with} \sum_{i,j} |c_{ij}|^2 = 1,$$

where $\{|e_{r,1}\rangle, |e_{r,2}\rangle, \ldots\}$ is an orthonormal basis for $\mathcal{H}_r$ with $r \in \{1, 2\}$.

A state of the form $|x\rangle = |x_1\rangle \otimes |x_2\rangle$ is a separable state or a tensor product state. Otherwise, it is an entangled state.

Theorem Every state $|x\rangle$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ admits a Schmidt decomposition

$$|x\rangle = \sum_{j=1}^{r} \sqrt{s_j} |u_j\rangle \otimes |v_j\rangle, \longrightarrow r \leq \min\{\dim \mathcal{H}_1, \dim \mathcal{H}_2\} = \min\{m, n\}$$

where $s_j > 0$ are the Schmidt coefficients satisfying $\sum_{j=1}^{r} s_j = 1$, r is the Schmidt number of $|x\rangle$, $\{|u_1\rangle, \ldots, |u_r\rangle\}$ is an orthonormal set of $\mathcal{H}_1$ and $\{|v_1\rangle, \ldots, |v_r\rangle\}$ is an orthonormal set of $\mathcal{H}_2$.

Remark Extending the results to $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ for $k \geq 3$ is an open problem.

Bipartite System

$$|u\rangle = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad |v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \Bigg\} \quad |u\rangle \otimes |v\rangle = \begin{pmatrix} u_1 v_1 \\ u_1 v_2 \\ u_2 v_1 \\ u_2 v_2 \end{pmatrix}$$

More generally

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} u_1 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ u_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ u_3 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ u_4 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \end{pmatrix}$$

Example.

$$|\psi\rangle = \frac{1}{\sqrt{1+4+9+16}} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \neq |u\rangle \otimes |v\rangle$$

$$= \sqrt{s_1}\, |u_1\rangle \otimes |v_1\rangle + \sqrt{s_2}\, |u_2\rangle \otimes |v_2\rangle \qquad s_1 + s_2 = 1$$

$$\in \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2}_{} \overset{\wedge}{\simeq} \mathbb{C}^4$$

$\shortparallel$

set of all linear combinations of $x \otimes y$, $x \in \mathbb{C}^2$, $y \in \mathbb{C}^2$.

$$\{e_{1,1}, e_{1,2}\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \text{ for } \mathcal{H}_1$$

$$\{e_{2,1}, e_{2,2}\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \text{ for } \mathcal{H}_2$$

$$\{ e_{1,j} \otimes e_{2,k} : 1 \le j, k \le 2 \}$$

$$= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$ is a basis for $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$

Schmidt decomposition = SVD.

Let $|x\rangle = \sum_{i,j} c_{ij} |e_{1,i}\rangle \otimes |e_{1,j}\rangle.$

where $\{|e_{1,1}\rangle, \dots, |e_{1,m}\rangle\}$ is a basis for $\mathcal{H}_1 \cong \mathbb{C}^m$

$\{|e_{2,1}\rangle, \dots, |e_{2,n}\rangle\}$ is a basis for $\mathcal{H}_2 \cong \mathbb{C}^n$.

Consider $m \times n$ matrix

$$[c_{ij}] = U \overbrace{\left[\begin{matrix} \sqrt{s_1} & 0 \\ 0 & \sqrt{s_m} \end{matrix} \,\middle|\, 0 \right]}^{n} V^+$$

$$= \left[u_1 \middle| \cdots \middle| u_m \right] \begin{bmatrix} \sqrt{s_1} & 0 \\ 0 & \sqrt{s_m} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$$

$=$

Then $|x\rangle = \sqrt{s_1}|u_1\rangle \otimes |v_1\rangle + \cdots + \sqrt{s_n}|u_m\rangle \otimes |v_m\rangle$

Example.

$$C = \frac{1}{2}\begin{pmatrix} 1 & 0 & i \\ 1 & 0 & i \end{pmatrix}$$

$|x\rangle = \frac{1}{2}(|e_{1,1}\rangle \otimes |e_{2,1}\rangle + |e_{1,1}\rangle \otimes |e_{2,3}\rangle + |e_{1,2}\rangle \otimes |e_{2,1}\rangle + |e_{1,2}\rangle \otimes |e_{2,3}\rangle)$

$$= \frac{1}{2} \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\sqrt{2}} (1) \frac{\begin{pmatrix} 1 & 0 & i \end{pmatrix}}{\sqrt{2}}.$$

$$|x\rangle = \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\sqrt{2}} \otimes \frac{\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}}{\sqrt{2}}.$$