Hint. for Ex 2.1.

$$\underset{A}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}\underset{B}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$ is not Hermitian

(1) $\langle\psi|AB|\psi\rangle = z_1$

$$\bar{z}_1 = \langle\psi|AB|\psi\rangle^{\dagger} = \langle\psi|B^{\dagger}A^{\dagger}|\psi\rangle = \langle\psi|BA|\psi\rangle = z_2$$

$$L.S. = |\langle\psi|AB-BA|\psi\rangle|^2 + |\langle\psi|AB+BA|\psi\rangle|^2$$
$$= |z_1 - z_2|^2 + |z_1 + z_2|^2$$
$$= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) + (z \ldots$$
$$= 4|z_1|^2$$

$$(x^*(AB)x)^*$$
$$= x^* B^* A^* x$$

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(2) Consider $tA + e^{i\theta}B$

$$0 \le \langle\psi|\underbrace{(tA + e^{i\theta}B)^{\dagger}}_{n\times n}\underbrace{(tA + e^{i\theta}B)|\psi\rangle}_{n\times n}$$ $|\psi\rangle$

$$= t^2\langle\psi|A^2|\psi\rangle + t\langle\psi|e^{i\theta}AB|\psi\rangle$$
$$+ t\langle\psi|e^{-i\theta}BA|\psi\rangle$$
$$+ \langle\psi|B^2|\psi\rangle$$

Choose θ so that $\langle\psi|e^{i\theta}AB|\psi\rangle$
$$= e^{i\theta}\langle\psi|AB|\psi\rangle$$
$$= -|\langle\psi|AB|\psi\rangle|$$

$$\therefore \langle\psi|e^{-i\theta}BA|\psi\rangle = \left(e^{i\theta}\langle\psi|AB|\psi\rangle\right)^{*} = \underset{b}{\underset{\shortparallel}{-|\langle\psi|AB|\psi\rangle|}}$$

$$0 \le \underset{\langle\psi|A^2|\psi\rangle}{\underset{\shortparallel}{t^2 \cdot a}} - t2\underset{|\langle\psi|AB|\psi\rangle|}{\underset{\shortparallel}{b}} + \underset{\langle\psi|B^2|\psi\rangle}{\underset{\shortparallel}{c}} \Rightarrow b^2 \le ac \ldots$$

$$0 \le \langle \xi | \xi \rangle = \|\xi\|^2$$

$$0 \le \langle \psi | (A^\dagger + B^\dagger) \boxed{(A+B)|\psi\rangle}$$

$$= \langle \psi | A^\dagger A + A^\dagger B + B^\dagger A + B^\dagger B | \psi \rangle$$

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(4) Let $\hat{A} = A - \langle A \rangle I$

$\hat{B} = B - \langle B \rangle I$.

Then $[\hat{A}, \hat{B}] = [A, B]$.

$\Delta(A) = \sqrt{\langle \hat{A}^2 \rangle}$

$\Delta(B) = \sqrt{\langle \hat{B}^2 \rangle}$

and apply (3) .

Remark: States $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$

$\mathcal{H}_1 \otimes \mathcal{H}_2$

$\mathbb{C}^m \otimes \mathbb{C}^n$

$$|u\rangle = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \quad |v\rangle = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$|u\rangle \otimes |v\rangle = |uv\rangle = \begin{pmatrix} u_1 v \\ u_2 v \\ \vdots \\ u_m v \end{pmatrix} \in \underset{\mathbb{C}^m \otimes \mathbb{C}^n}{\underset{\parallel}{\mathbb{C}^{mn}}}$$

In general, $|x\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ is an vector state in $\mathbb{C}^{mn} \simeq \underset{\uparrow}{\mathbb{C}^m \otimes \mathbb{C}^n}$

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the vector space of all linear combinations of $|u\rangle \otimes |v\rangle$ with $|u\rangle \in \mathbb{C}^n$, $|v\rangle \in \mathbb{C}^n$.

In fact, if $\{e_{1,1}, \dots, e_{1,m}\}$ is a basis for $\mathbb{C}^m$

$\{e_{2,1}, \dots, e_{2,n}\}$ is a basis for $\mathbb{C}^n$

then $\left\{ e_{1,j} \otimes e_{2,k} : \begin{matrix} 1 \le j \le m \\ 1 \le k \le n \end{matrix} \right\}$ is a basis for $\underset{\mathbb{C}^{mn}.}{\underset{\parallel}{\mathbb{C}^m \otimes \mathbb{C}^n}}$

Example

$\underline{\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}}$ a basis for $\mathbb{C}^2$

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is basis for $\mathbb{C}^2$

Then $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is

a basis for $\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$.

$$|u\rangle = \sum_{i=1}^{m} u_i|e_{1,i}\rangle, \quad |v\rangle = \sum_{j=1}^{n} v_j\,|e_{2,j}\rangle$$

$$|u\rangle\otimes|v\rangle = \sum_{\substack{1\le i\le m\\ 1\le j\le n}} u_i v_j\, |e_{1,i}\rangle\otimes|e_{2,j}\rangle .$$

$$\begin{pmatrix} u_1 \boxed{v} \\ \hline u_2 \boxed{v} \\ \hline u_3 \boxed{v} \\ \vdots \\ u_m \boxed{v} \end{pmatrix} \begin{matrix} \}n \\ \}n \\ \}n \\ \\ \}n \end{matrix}$$

Schmidt decomposition:

$$|x\rangle = \sum c_{ij}\, |e_{1,i}\rangle\otimes|e_{2,j}\rangle \quad \text{with} \quad \sum|c_{ij}|^2 = 1$$

$$= \sum_{l=1}^{r} \sqrt{s_l}\, |u_l\rangle\otimes|v_l\rangle$$

$$\begin{pmatrix} c_{11} \\ c_{12} \\ \hline c_{21} \\ c_{22} \end{pmatrix}$$

where $r \le \min\{m,n\}$,

$\{|u_1\rangle, \dots, |u_l\rangle\}$ is an orthormal set in $\mathbb{C}^m$

$\{|v_1\rangle, \dots, |v_l\rangle\}$ is an orthormal set in $\mathbb{C}^n$

$s_1 + \dots + s_l = 1$.

Proof: Write (c_{ij}) as an $m\times n$ matrix, as so that

$$(u_{ij}) \left(\begin{array}{ccc|c} \sqrt{s_1} & & 0 & \\ & \ddots & & 0 \\ 0 & & \sqrt{s_l} & \\ \hline & 0 & & \underbrace{0}_{n-l} \end{array}\right) \}_{m-l} \, (v_{ij})$$

$$= \begin{pmatrix} u_{11} \\ \vdots \\ u_{m1} \end{pmatrix} \sqrt{s_1}\, (v_{11} \cdots v_{1n}) + \cdots + \begin{pmatrix} u_{1l} \\ \vdots \\ u_{ml} \end{pmatrix} \sqrt{s_l}\, (v_{l1} \cdots v_{ln})$$

Can be translated back to

$$\Longrightarrow \quad |x\rangle = \sqrt{s_1}\, |u_1\rangle\otimes|v_1\rangle + \cdots + \sqrt{s_l}\, |u_l\rangle\otimes|v_l\rangle$$

Remark:

$$\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^p$$

$$|x\rangle = \sum_{\substack{1\le i\le m\\ 1\le j\le n\\ 1\le j\le p}} c_{ijk}\, |e_{1,i}\rangle \otimes |e_{2,j}\rangle \otimes |e_{3,k}\rangle$$

$$\neq \quad s_1\, |u_1\rangle \otimes |v_1\rangle \otimes |w_1\rangle$$

$$+ \vdots$$

$$+ s_\ell\, |u_\ell\rangle \otimes |v_\ell\rangle \otimes |w_\ell\rangle .$$

Mixed States and Density Matrices

A system is in a mixed state if there is a probability p_i that the system is in state $|x_i\rangle$ for $i = 1, \ldots, N$. If there is only one possible state, i.e., $p_1 = 1$, then the system is in pure state. The mean value of the measurement of the system corresponding to the observable described by the Hermitian matrix A is

$m \times n \quad n \times m \qquad n \times m \quad m \times n$

$\mathrm{tr}\, \underline{XY} = \mathrm{tr}\, \underline{YX}$

$$\langle A \rangle = \underline{\sum_{j=1}^{N} p_i \langle x_j | A | x_j \rangle} = \mathrm{tr}\,(A\rho), =$$

$$\to \sum p_i \langle x_j | A x_j \rangle = \mathrm{tr} \sum p_i A |x_i\rangle\langle x_i|$$

$$= \mathrm{tr}\, A \left(\sum p_i |x_i\rangle\langle x_i| \right.$$

where

$$\boxed{\rho = \sum_{j=1}^{N} p_j |x_j\rangle\langle x_j|}$$

is a density operator (matrix).

Example.

$$x_1 : \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\sqrt{2}} \qquad p_1 = \frac{1}{3}$$

$$x_2 : \frac{\begin{pmatrix} \sqrt{3} \\ i \end{pmatrix}}{2} \qquad p_2 = \frac{2}{3}$$

In terms of measurement, using observable corresponding to the Hermitian A, we get the measurement for the mixed states system p_i, $|\psi_i\rangle$ $i = 1, \ldots, N$.

$$\sum p_i \langle \psi_i | A | \psi_i \rangle = 1$$

Description of a quantum system in mixed states.

A1' A physical state is specified by a density matrix $\rho : \mathcal{H} \to \mathcal{H}$, which is positive semidefinite with trace equal to one.

A2' The mean value of an observable associate with the Hermitian matrix A is $\langle A \rangle = \mathrm{tr}\,(\rho A)$.

$\langle x | A | x \rangle$

A3' The temporal evolution of the density matrix is given by the Liouville-von Neumann equation

$[\rho_{ij}(t)]$

$$i\hbar \frac{d}{dt}\rho = [H, \rho] = H\rho - \rho H,$$

$\frac{d}{dx}(|x\rangle\langle x|) =$

where H is the system Hamiltonian.

Theorem 2.1 The following conditions are equivalent for a given state (density matrix) ρ.

(a) ρ is pure. (b) $\rho^2 = \rho$. (c) $\mathrm{tr}\,(\rho^2) = 1$.

Definition 2.1 Suppose $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. A state ρ is **uncorrelated** if $\rho = \rho_1 \otimes \rho_2$; it is **separable** if it is a convex combination of uncorrelated states, i.e.,

$$\rho = \sum_{j=1}^{r} p_j \rho_{1,j} \otimes \rho_{2,j}.$$

Otherwise, it is **inseparable**.