$$\downarrow \quad \downarrow$$

$|00\rangle$
$|01\rangle$
$|10\rangle$
$|11\rangle$

4.6 Universal quantum gates

Theorem The set of $\boxed{\text{single qubit gates}}$ and the $\boxed{\text{CNOT}}$ gate from a universal set.

Example for 2-qubit gates.

$$U = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} = U_1 U_2 U_3 U_4 U_5 U_6, \quad \text{where } U_i \in \{ [\ldots], [\ldots], [\ldots] \}$$

two level unitary

In general, we use the grey code to realize a two-level unitary gate as a the product of CNOT gates and a controlled qubit gate.

Every controlled qubit gate is a product of at most two CNOT gates and three single qubit gates. (Lemma 4.4)

Consider $\mathcal{U} = (u_{ij}) \in M_4$ is unitary

Let $V_1 = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & a & b \\ 0 & & c & d \end{bmatrix}$ be unitary such that

$$V_1 U = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \end{bmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u_{31} \\ u_{41} \end{pmatrix} = \begin{pmatrix} \sqrt{|u_{31}|^2+|u_{41}|^2} \\ 0 \end{pmatrix}$$

i.e., $\dfrac{1}{\sqrt{|u_{31}|^2+|u_{41}|^2}}\begin{pmatrix} \bar{u}_{31} & \bar{u}_{41} \\ -u_{41} & u_{31} \end{pmatrix}$

Then $V_2 V_1 U = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix}$

$$\begin{pmatrix} 1 & & & \\ & a' & b' & \\ & c' & d' & \\ & & & 1 \end{pmatrix}$$

$u_{31}\cdot(-u_{41}) + u_{41}u_{31} = 0$

Then $V_3 V_2 V_1 U$ $\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix}$

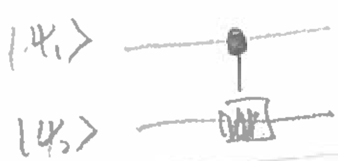
Then $V_4 V_3 V_2 V_1 U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix}$

Then $V_5 V_4 \cdots U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}$

Then $V_6 \cdots V_1 U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$

$\therefore U = V_1^\dagger \cdots V_5^\dagger V_6^\dagger$

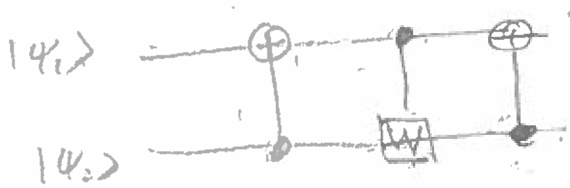
$$\underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & \begin{smallmatrix} x & x \\ x & x \end{smallmatrix} \end{bmatrix}}_{W}\left(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle\right)$$
$$= \left(a|00\rangle + b|01\rangle + \hat{c}|10\rangle + \hat{d}|11\rangle\right) \qquad \begin{pmatrix} \hat{c} \\ \hat{d} \end{pmatrix} = W\begin{pmatrix} c \\ d \end{pmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & x & 0 \\ 0 & x & x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\left(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle\right) = a|00\rangle + \hat{b}|01\rangle + \hat{c}|10\rangle + d|11\rangle$$

$(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle)|1\rangle \mapsto$

$(a|00\rangle + b|11\rangle + c|10\rangle + d|01\rangle)$



$P^\dagger(\ldots)P(\ldots)$

Grey Code:

$|00\rangle =$
$|01\rangle =$
$|10\rangle =$
$|11\rangle =$

$|0\rangle|0\rangle|0\rangle$
$|0\rangle|0\rangle|1\rangle$
$|0\rangle|1\rangle|1\rangle$
$|0\rangle|1\rangle|0\rangle$
$|1\rangle|1\rangle|0\rangle$
$|1\rangle|1\rangle|1\rangle$
$|1\rangle|0\rangle|1\rangle$
$|1\rangle|0\rangle|0\rangle$

So we can always transform $|\tilde{\imath}_1 \cdots \tilde{\imath}_n\rangle$ to $|\hat{\imath}_1 \cdots \hat{\imath}_n\rangle$ differing only in one position of $|j_1, \cdots, j_n\rangle$ for any $|j_1, \cdots, j_n\rangle$.

Hence for any two level unitary U acting on the positions labeled by $|\tilde{\imath}_1 \cdots \tilde{\imath}_n\rangle$, $|j_1 \cdots j_n\rangle$ we can change the vector states so that U will act on one of the qubits. So we can apply the 2×2 unitary W to the single qubit using all other qubits as control bits