Math 410 Intro to Quantum Computing Homework 10

Sample Solution

10.2 (a) $U_H Z U_H = (\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix})\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}) = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = X.$

(b) $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) - |-\rangle$; $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - (-|1\rangle)) = |+\rangle$;

(c) $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$; $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle) = -|-\rangle$;

10.3 Note that $e^{i\beta Z} = \cos\beta I + i\sin\beta Z$, which is a linear combination of I and Z. So, the linear span of $E = \{IIZ, IZI, IIZ\}$ is the same as that of $E' = \{IIU_\beta, IU_\beta IU_\beta II\}$. Hence, the QECC for channel with error operators in E also works for error operators in E'.

Alternatively, suppose U_β occurs in the first qubit. Then $(U_\beta \otimes I \otimes I)(a|+++\rangle + b|---\rangle)$

$= a(\cos\beta|+++\rangle + i\sin\beta|-++\rangle) + b(\cos\beta|---\rangle + \sin\beta|+--\rangle)$.

The Hadamard gates in the error syndrome detection circuit maps these vectors as

$$a(\cos\beta|000\rangle + i\sin\beta|100\rangle) + b(\cos\beta|111\rangle + i\sin\beta|011).$$

The error syndrome detection circuit outputs $\cos\beta(a|000\rangle + b|111\rangle)|00\rangle + i\sin\beta(a|100\rangle + b|011\rangle)|11\rangle$. The receiver gets $a|000\rangle + b|111\rangle$ when they the measure ancilla qubits 00 and 11 with probabilities $\cos^2\beta$ and $\sin^2\beta$, respectively. So, the circuit in Figure 10.4 can correct the error, and the Hadamard gates will give the correct transmitted state.

10.4 If $|\psi\rangle = |0\rangle$, there is $\frac{1}{\sqrt{2}}$ chance after the Hadamard gate it is mapped into $|0\rangle$, and after tripilating the system becomes $|000\rangle$; there is $\frac{1}{\sqrt{2}}$ chance after the Hadamard gate it is mapped into $|1\rangle$, and after tripilating the system becomes $|111\rangle$. Therefore, $|0\rangle|00\rangle \to \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) = |+\rangle$

If $|\psi\rangle = |1\rangle$, there is $\frac{1}{\sqrt{2}}$ chance after the Hadamard gate it is mapped into $|0\rangle$, and after tripilating the system becomes $|000\rangle$; there is $\frac{1}{\sqrt{2}}$ chance after the Hadamard gate it is mapped into $-|1\rangle$, and after tripilating the system becomes $-|111\rangle$. Therefore, $|1\rangle|00\rangle \to \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle) = |-\rangle$.

10.5 Suppose the first qubit is flipped, $x_1' = x_1 \oplus 1$; so $x_1' \oplus x_2 = 1 \oplus 0 = 1, x_1' \oplus x_3 = 1 \oplus 0 = 1$. That is, $A_1 = 1, B_1 = 1$.

Suppose the second qubit is flipped, $x_2' = x_2 \oplus 1$; so $x_1 \oplus x_2' = 0 \oplus 1 = 1, x_1 \oplus x_3 = 0$. That is, $A_1 = 1, B_1 = 0$.

Suppose the third qubit is flipped, $x_3' = x_3 \oplus 1$; so $x_1 \oplus x_2 = 1, x_1 \oplus x_3' = 0 \oplus 1 = 1$. That is, $A_1 = 1, B_1 = 1$.

10.6 When there is no error, $x_1 \oplus x_2 = x_1 \oplus x_3 = 0$

Because there is no error in the second and third group, $A_2 = B_2 = 0, A_3 = B_3 = 0$.

When a phase error exists in the second qubit $x_2' = -x_2$. We see $x_2 - x_2' = 0 \pmod 2$. $x_1 \oplus x_2' = 0 - 0 \pmod 2 = 0$. And because x_3 is unchanged, we still have $x_1 \oplus x_3 = 0$.

Therefore, $A_1 = B_1 = 0$.

10.7 (1) Because there is no error in the first and second qubit, $A_1 = B_1 = 0, A_2 = B_2 = 0$. As we have seen in part 10.6, a phase shift does not vary A_i, B_i, effect of a Y error on A_3, B_3 is equivalent to an X error on them. As found in part 10.5, $A_3 = B_3 = 1$.

(2) Applying σ_x to first qubit of the third group, we obtain $a|++-\rangle + b|--+\rangle$. From table 10.3, $(A_4, B_4) = (0, 1)$.

(3) Note that $Y = ZX$. So,

$$(U_\beta \otimes I \otimes I)(a|000\rangle + b|111\rangle) = \cos\beta(a|000\rangle + b|111\rangle) + i\sin\beta(a|000\rangle - b|011\rangle), \text{ and}$$

$$(U_\alpha \otimes I \otimes I)(a|000\rangle + b|111\rangle)(\cos\beta(a|000\rangle + b|111\rangle) + i\sin\beta(a|000\rangle - b|011\rangle))$$

$$= \cos(\alpha+\beta)(a|000\rangle + b|111\rangle) + i\sin(\alpha+\beta)(a|100\rangle + b|011\rangle).$$

Thus, we have $(A_4, B_4) = (1, 1)$ after applying the first layer of the Hadamard gates.

10.8 Since $H[0001110]^t = [3 \pmod 2, 1, 1]^T = [1, 1, 1]^t$, $H[1101000]^t = [1, 1, 1]^t$, and

$$H[1100111]^t = [3 \pmod 2, 3 \pmod 2, 3 \pmod 2]^t = [1, 1, 1]^t,$$

we apply a bit flip at the seventh bit in each case to do the correction.

10.9 (a) Since $d(c_1, c_2)$ is the number of different digits in c_1, c_2, $d(c_1, c_2) \geq 0$.

(b) Since the number of digits in c_2 differing from c_1 is the same as the number of digits in c_1 differing from c_2, we see that $d(c_1, c_2) = d(c_2, c_1)$.

(c) Let $c_1 = (x_1, \ldots, x_n), c_2 = (y_1, \ldots, y_n), c_3 = (z_1, \ldots, z_n)$. Define $|u - v| = 0$ if $u = v$ and $|u - v| = 1$ if $u \neq v$. Then $d(c_1, c_2) = \sum_j |x_j - y_j|$, $d(c_2, c_3) = \sum_j |y_j - z_j|$. It is clear that for each j, $|x_j - z_j| \leq |x_j - y_j| + |y_j - z_j|$. So, $d(c_1, c_3) = \sum_j |x_j - z_j| \leq d(c_1, c_2) + d(c_2, c_3)$.

10.10 Note that $d(c_1, c_2) = d(c_1 - c_1, c_2 - c_1)$ because both sides equal the numbers of entries in c_1 differing from c_2. So, $\min\{d(c_1, c_2) : c_1 \neq c_2\}$ is the same as $\min\{d(0, c) : c \neq 0\}$. One readily see that the 15 nonzero codewords has at least 3 nonzero entries. The result follows.

10.11 (1) Note that $X^2 = Z^2 = I_2$. Thus, $M_i^2 = I$ for $i = 0, \ldots, 4$. Because M_i is a tensor product of I_2, X, Z and all these matrices have eigenvalues in $\{1, -1\}$, the eigenvalues of M_i will be product of numbers in $\{1, -1\}$ and will lie in the set $\{1, -1\}$.

(2) Use the fact that $XZ = -ZX$, $X^2 = Z^2 = I$. We have

$$[M_0, M_1] = Z \otimes Z \otimes XZ \otimes X^2 \otimes ZX - Z \otimes Z \otimes ZX \otimes X^2 \otimes XZ = 0,$$

$$[M_0, M_2] = X \otimes Z^2 \otimes X \otimes XZ \otimes ZX - X \otimes Z^2 \otimes X \otimes ZX \otimes XZ = 0,$$

$$[M_0, M_3] = X \otimes ZX \otimes XZ \otimes X \otimes Z^2 - X \otimes XZ \otimes ZX \otimes X \otimes Z^2 = 0,$$

$$[M_1, M_2] = ZX \otimes Z \otimes Z \otimes XZ \otimes X^2 - XZ \otimes Z \otimes Z \otimes ZX \otimes X^2 = 0,$$

$$[M_1, M_3] = ZX \otimes X \otimes Z^2 \otimes X \otimes XZ - XZ \otimes X \otimes Z^2 \otimes X \otimes ZX = 0,$$

$$[M_2, M_3] = X^2 \otimes ZX \otimes Z \otimes Z \otimes XZ - X^2 \otimes XZ \otimes Z \otimes Z \otimes ZX = 0.$$

(3) $M_4 = M_0 M_1 M_2 M_3 = (X_2 X_3 Z_1 Z_4)(X_3 X_4 Z_2 Z_0)(X_4 X_0 Z_3 Z_1)(X_0 X_1 Z_4 Z_2)$

$$= X_2 X_3^2 Z_1 Z_4 X_4^2 Z_2 Z_0 X_0^2 Z_3 Z_1 X_1 Z_4 Z_2 = X_2 Z_1^2 Z_4^2 Z_2^2 Z_0 Z_3 Z_1 = X_1 X_2 Z_0 Z_3.$$