Math 410 Intro to Quantum Computing Homework 5

Sample Solution

2.7 Note $|\Psi'\rangle = \frac{1}{2}(|e_1\rangle \otimes |e_2\rangle - |e_2\rangle \otimes |e_1\rangle) = \frac{1}{\sqrt{2}}(0,1,-1,0)^t$. Thus,

$$\rho' = |\Psi'\rangle\langle\Psi'| = \frac{1}{2}(0,1,-1,0)^t(0,1,-1,0) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathrm{Tr}_1(\rho')$ is the sum of the leading $(1,1)$ block and $(2,2)$ block, and equals $I/2$.

2.8 Spectral decompose ρ, we have $\rho = \frac{1}{4}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + \frac{3}{4}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}$.

We may let $|\Psi\rangle = \frac{1}{2}|0\rangle \otimes |e_1\rangle + \frac{\sqrt{3}}{2}|1\rangle \otimes |e_2\rangle = \frac{1}{2}(1,0,0,\sqrt{3})^t$, where $\{|e_1\rangle, |e_2\rangle\} = \{(1,0)^t, (0,1)^t\}$. So, $\hat{\rho} = |\Psi\rangle\langle\Psi| = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 3 \end{pmatrix}$ and $\mathrm{Tr}_2(\hat{\rho})$ is the matrices of traces of the blocks of the matrix $\hat{\rho}$ and equals ρ.

2.9 Suppose $\mathcal{H}$ has dimension n. If $|\Psi\rangle = \sum_k \sqrt{p_k}|\psi_k\rangle|\phi_k\rangle \in \mathcal{H} \otimes \mathcal{H}$ is such that $\mathrm{Tr}_2(|\Psi\rangle\langle\Psi|) = \rho$, then there is an orthonormal basis $\{|\alpha_j\rangle : 1 \le j \le n\}$ for $\mathcal{H}$ such that $\mathrm{Tr}_2(|\Psi\rangle\langle\Psi|) = \rho$. Then for any unitary U and $|\Psi'\rangle = \sum_k \sqrt{p_k}|\psi_k\rangle U|\phi_k\rangle$, we can consider the orthonormal basis $\{U|\alpha_j\rangle : 1 \le j \le n\}$ for $\mathcal{H}$ such that

$$\mathrm{Tr}_2(|\Psi'\rangle\langle\Psi'|) = \sum_j (I \otimes \langle\alpha_j|U^\dagger)(\sum_k(|\psi\rangle \otimes U|\phi_j\rangle))(\sum_k(\langle\psi| \otimes \langle\phi_j|U^\dagger)(\sum_j(I \otimes U|\alpha_j\rangle))$$

$$= \sum_j (I \otimes \langle\alpha_j|)(\sum_k(|\psi\rangle \otimes |\phi_j\rangle))(\sum_k(\langle\psi| \otimes \langle\phi_j|)(\sum_j(I \otimes |\alpha_j\rangle)) = \sum_j (I \otimes \langle\alpha_j|)(|\Psi\rangle\langle\Psi|)(I \otimes |\alpha_j\rangle) = \rho.$$

Note Here we cannot assume that $\{|\phi_1\rangle, |\phi_2\rangle, \ldots\}$ is an orthonormal basis.

2.10 Note that for any positive operator A, $\sqrt{UAU^\dagger} = U\sqrt{A}U^\dagger$ because $B = U\sqrt{A}U^\dagger$ satisfies $B^2 = UAU^\dagger$, and $\mathrm{Tr}(XY) = \sum_{i,j} x_{ij}y_{ji} = \mathrm{Tr}(YX)$ for $X = (x_{ij})$ and $Y = (y_{ij})$.

Now suppose ρ_1, ρ_2 are positive operators. Then

$$F(U\rho_1U^\dagger, U\rho_2U^\dagger) = \mathrm{Tr}\sqrt{\sqrt{U\rho_1U^\dagger}U\rho_2U^\dagger\sqrt{U\rho_1U^\dagger}} = \mathrm{Tr}\sqrt{U\sqrt{\rho_1}\rho_2\sqrt{\rho_1}U^\dagger}$$

$$= \mathrm{Tr}(U\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}U^\dagger) = \mathrm{Tr}(\sqrt{\rho_1}\rho_2\sqrt{\rho_1}U^\dagger U) = \mathrm{Tr}(\sqrt{\rho_1}\rho_2\sqrt{\rho_1}) = F(\rho_1, \rho_2).$$

2.11 Note that $\rho_1 = \frac{1}{2}[1\ 0\ 0\ 0]^T[1\ 0\ 0\ 0] + \frac{1}{2}[0\ 0\ 0\ 1]^T[0\ 0\ 0\ 1]$ and $\sqrt{\rho_1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}$. Thus,

$\sqrt{\rho_1}\rho_2\sqrt{\rho_1} = \frac{1}{4}\begin{bmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{bmatrix} = \frac{1}{2}\cdot(\frac{1}{\sqrt{2}})^2[1\,0\,0\,1]^T[1\,0\,0\,1]$, so $\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}} = \sqrt{\frac{1}{2}}\cdot\frac{1}{2}[1\,0\,0\,1]^T[1\,0\,0\,1] = \frac{1}{2\sqrt{2}}\begin{bmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{bmatrix}$. Therefore, $F(\rho_1,\rho_2) = \frac{1}{2\sqrt{2}} + 0 + 0 + \frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$.

3.1 A direct computation shows that

$$\begin{bmatrix}\cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\end{bmatrix}\sigma_x\begin{bmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{bmatrix} = e^{-i\theta}\cos\frac{\phi}{2}\sin\frac{\phi}{2} + e^{i\theta}\cos\frac{\phi}{2}\sin\frac{\phi}{2} = 2\cos\theta\cos\frac{\phi}{2}\sin\frac{\phi}{2} = \cos\theta\sin\phi;$$

$$\begin{bmatrix}\cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\end{bmatrix}\sigma_y\begin{bmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{bmatrix} = ie^{-i\theta}\cos\frac{\phi}{2}\sin\frac{\phi}{2} - ie^{i\theta}\cos\frac{\phi}{2}\sin\frac{\phi}{2} = 2\sin\theta\cos\frac{\phi}{2}\sin\frac{\phi}{2} = \sin\theta\sin\phi;$$

$$\begin{bmatrix}\cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\end{bmatrix}\sigma_z\begin{bmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{bmatrix} = \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2} = \cos\phi.$$

Hence, $\langle\psi(\theta,\phi)|\sigma|\psi(\theta,\phi)\rangle = [\cos\phi\sin\theta\ \ \sin\phi\sin\theta\ \ \cos\phi]^{\mathbf{T}} = \hat{\mathbf{n}}(\theta,\phi)$.

3.3 If $\rho = \frac{1}{2}\begin{bmatrix}1+u_z & u_x - iu_y\\ u_x + iu_y & 1-u_z\end{bmatrix}$, then

$$\rho\sigma_x = \tfrac{1}{2}(u_x - iu_y + u_x + iu_y) = u_x,\ \rho\sigma_y = \tfrac{1}{2}(iu_x + u_y - iu_x + u_y) = u_y,\ \rho\sigma_z = \tfrac{1}{2}(1 + u_z - 1 + u_z) = u_z.$$

Therefore, $\langle\sigma\rangle = \mathrm{Tr}(\rho\sigma) = \mathbf{u}$.

3.5 Note that $\sigma_x \otimes \sigma_z = \begin{bmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{bmatrix}$. A direct computation shows that

$$\langle\Phi^+|(\sigma_x\otimes\sigma_z)|\Phi^+\rangle = 0,\ \langle\Phi^-|(\sigma_x\otimes\sigma_z)|\Phi^-\rangle = 0,\ \langle\Psi^+|(\sigma_x\otimes\sigma_z)|\Psi^+\rangle = 0,\ \langle\Psi^-|(\sigma_x\otimes\sigma_z)|\Psi^-\rangle = 0.$$