Some macros for typesetting quantum computing homework **C.K. Li**

- In quantum physics, we use the unit vectors $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to represent the states.

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1\ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = E_{11} \quad \text{and} \quad |1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0\ 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = E_{22}.$$

- A photon in a quantum environment has the form $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ with $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. Note that we have to use complex numbers!
- In matrix form, we have $\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |a|^2 & a\bar{b} \\ b\bar{a} & |b|^2 \end{pmatrix}$. Note that $\rho = \rho^\dagger$.
- Example: If $|\psi\rangle = \sqrt{3} \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$, then $|\psi\rangle\langle\psi|$.
- Suppose $A = E_{11}, B = E_{22}, C = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $ABC = 0 = CAB$, but $ACB \neq 0$.
- Note that $C = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $D = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ are two measurable states of photons using a different basis/frames.
- $\mathbb{C}^n$ is a vector space under addition and multiplication.

  $v_1 + v_2$ for $v_1, v_2 \in \mathbb{C}^n$, μv for $\mu \in \mathbb{C}, v \in \mathbb{C}^n$.
- In particular, $\langle v|$ is the conjugate transpose of $|v\rangle$.
- It is easy to express a vector as a linear combination of orthonormal basis $\{|e_1\rangle, \dots, |e_n\rangle\}$.
- The set $\{P_j = |e_j\rangle\langle e_j| : j = 1, \dots, n\}$ form a complete set of projection operators/matrices.
- Gram-Schmidt orthogonalization/orthonormalization.

  Let $\{|x_1\rangle, \dots, |x_m\rangle\}$ be linearly independent.

  Set $|e_1\rangle = |x_1\rangle / \| |x_1\rangle \|$.

  Set $|e_2\rangle = |f_2\rangle / \| |f_2\rangle \|$, where $|f_2\rangle = |x_2\rangle - \langle e_1|x_2\rangle |e_1\rangle$ is orthogonal to $|e_1\rangle$.

  For $k > 1$, set $|f_k\rangle / \| |f_k\rangle \|$, where $|f_k\rangle = |x_k\rangle - (\sum_{j=1}^{k-1} \langle e_j|x_k\rangle |e_j\rangle$ is orthogonal to $|e_1\rangle, \dots, |e_k\rangle$.
- For any basis $\{|x_1\rangle, \dots, |x_n\rangle\}$ of $\mathbb{C}^n$, there is an orthonormal basis $\{|e_1\rangle, \dots, |e_n\rangle\}$ such that $\{|x_1\rangle, \dots, |x_m\rangle\}$ and $\{|e_1\rangle, \dots, |e_m\rangle\}$ span the same subspace for $m = 1, \dots, n$.
- For any linearly independent set $\{|x_1\rangle, \dots, |x_m\rangle\}$, we can get an orthonormal basis $\{|e_1\rangle, \dots, |e_n\rangle\}$ such that $\{|x_1\rangle, \dots, x_k\rangle\}$ and $\{|e_1\rangle, \dots, |e_k\rangle\}$ span the same subspace for $k = 1, \dots, m$.