# Linear maps preserving permutation and stochastic matrices

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Dedicated to Professor T. Ando

## Abstract

Let $\mathcal{S}$ be the set of $n \times n$ (sub)permutation matrices, doubly (sub)stochastic matrices, or the set of $m \times n$ column or row (sub)stochastic matrices. We characterize those linear maps $T$ on the linear span of $\mathcal{S}$ that satisfy $T(\mathcal{S}) = \mathcal{S}$. Partial results concerning those linear maps $T$ satisfying $T(\mathcal{S}) \subseteq \mathcal{S}$ are also presented.

*AMS classification*: 15A04, 15A51.
*Keywords*: Linear preserver; stochastic matrix; permutation matrix.

# 1 Introduction

Let $\mathcal{S}$ be a nonempty subset of a linear space $V$, and $T$ be a linear map on $V$. We say that $T$ *preserves* (respectively *strongly preserves*) $\mathcal{S}$ and that $T$ is a *linear preserver* (respectively *linear strong preserver*) of $\mathcal{S}$ if $T(\mathcal{S}) \subseteq \mathcal{S}$ (respectively $T(\mathcal{S}) = \mathcal{S}$). There has been a great deal of interest in studying linear maps preserving or strongly preserving special subsets; see [5] and its references. In this paper, we completely determine the structure of those linear maps $T$ on the linear space span($\mathcal{S}$) spanned by $\mathcal{S}$ satisfying $T(\mathcal{S}) = \mathcal{S}$ when $\mathcal{S}$ is one of the following sets:

**P**$(n)$: the group of $n \times n$ permutation matrices;

**DS**$(n)$: the convex set of $n \times n$ doubly stochastic matrices;

**sP**$(m, n)$: the set of $m \times n$ subpermutation matrices, i.e., zero-one matrices each of whose rows and columns has at most one nonzero entry;

**DsS**$(m, n)$: the set of $m \times n$ doubly substochastic matrices, i.e., nonnegative matrices with all row sums and column sums less than or equal to one;

**CS**$(m, n)$ (**RS**$(m, n)$): the set of $m \times n$ column (row) stochastic matrices, i.e., nonnegative matrices with all column (row) sums equal to one;

[1]Supported grants from USA NSF and the Reves Center of the College of William & Mary
[2]Supported by the National Science Council of the Republic of China

$\mathbf{CsS}(m,n)$ ($\mathbf{RsS}(m,n)$): the set of $m \times n$ column (row) substochastic matrices, i.e., nonnegative matrices with all column (row) sums less than or equal to one.

We will present these characterization theorems in Sections 2 and 3. In addition to enriching the list of linear preserver results, it is interesting to see how our proofs are done by a combination of techniques from various subjects including linear algebra, combinatorial theory, and theory of convex sets. In particular, our proofs depend much on knowledge about the geometry of faces of the convex sets $\mathbf{DS}(n), \mathbf{DsS}(n), \mathbf{CS}(m,n)$ and $\mathbf{CsS}(m,n)$.

The problem of characterizing those linear maps $T$ satisfying $T(\mathcal{S}) \subseteq \mathcal{S}$ with one of the above sets $\mathcal{S}$ is more intricate. We present some partial results on these problems in Section 4. Further remarks and related questions are presented in Section 5.

Note that there are related results in the literature. In [1, 8], the authors considered linear maps on complex matrices that map the set of complex matrices with all row sums and column sums equal to one into itself. In [2], the authors considered linear maps on complex matrices leaving invariant some special type of generalized permutation matrices. It is somewhat interesting to observe that so far no one has studied the natural question of real linear maps that map the set of doubly stochastic matrices (respectively, permutation matrices) into itself.

Below we give some definitions and results which we need in this paper.

Denote by $\{e_1, \dots, e_n\}$ the standard basis consisting of coordinate vectors for $\mathbb{R}^n$, and by $\{E_{11}, E_{12}, \dots, E_{mn}\}$ the standard basis for $\mathbb{R}^{m\times n}$. (When we use these symbols, the positive integers $m, n$ are either fixed or are understood from the context.) We also denote by $J_n$ the $n \times n$ matrix of all ones.

We say that a function $T : V \to V$ *fixes* an element $x \in V$ if $T(x) = x$. If $C$ is a nonempty subset of a finite dimensional linear space and $T$ is a linear map on span$(C)$ which strongly preserves $C$, then clearly $T$ is invertible and $T^{-1}$ also strongly preserves $C$.

For any permutation $\sigma$ in $S_n$, the symmetric group of degree $n$, we use $P_\sigma$ to denote the $n \times n$ permutation matrix whose $j$th column is $e_{\sigma(j)}$. For any distinct integers $i_1, \dots, i_k \in \{1, \dots, n\}$, we use $P(i_1, \dots, i_k)$ to denote the $n \times n$ permutation matrix $P_\sigma$, where $\sigma$ is the cycle given by $\sigma(j) = j$ for $j \notin \{i_1, \dots, i_k\}$, $\sigma(i_r) = i_{r+1}$ for $r = 1, \dots, k-1$, and $\sigma(i_k) = i_1$. It is easy to show that for any distinct integers $i_1, \dots, i_k \in \{1, \dots, n\}$ and any permutation $\sigma \in S_n$, we have $P_\sigma^t P(i_1, \dots, i_k) P_\sigma = P(\sigma^{-1}(i_1), \dots, \sigma^{-1}(i_k))$.

Let $C$ be a convex set in a finite dimensional real linear space. We use $\mathcal{E}(C)$ to denote the set of all extreme points of $C$. We call a subset $F$ of $C$ a *face* of $C$ if $F$ is convex and satisfies that if $(1-\lambda)x + \lambda y \in F$, where $0 < \lambda < 1$ and $x, y \in C$, then $x, y \in F$. For any $x \in C$, we denote by $\Phi(x : C)$ the face of $C$ generated by $x$, i.e., the intersection of all faces of $C$ containing $x$, or equivalently, the set $\{y \in C : x + \mu(x - y) \in C \text{ for some } \mu > 0\}$.

We shall assume elementary properties of a convex set. For reference, see [6]. In particular, if $C$ is a convex compact subset of a finite dimensional linear space $V$ and $T$ is a linear map on $V$, then $T(C) = C$ if and only if $T(\mathcal{E}(C)) = \mathcal{E}(C)$. The following are central to our proof of the characterization theorems of linear strong preservers in Sections 2 and 3.

**Proposition 1.1** *Let $C$ be a convex set in a finite dimensional real linear space $V$ and let $T$ be a linear map on $V$ satisfying $T(C) = C$. If $x \in C$, then $T(\Phi(x : C)) = \Phi(T(x) : C)$, and $T$ induces a bijection from $\Phi(x : C) \cap \mathcal{E}(C)$ to $\Phi(T(x) : C) \cap \mathcal{E}(C)$.*

By a *convex cone* in a finite dimensional real linear space $V$ we mean a nonempty subset $K$ which satisfies $\alpha x + \beta y \in K$ whenever $x,\ y \in K$ and $\alpha,\ \beta \geq 0$. Clearly a convex cone is also a convex set.

**Proposition 1.2** *Let $K$ be a convex cone in a finite dimensional real linear space $V$. Let $C$ be a convex set which consists of all elements in $K$ that satisfy a given finite collection of linear inequalities*

$$f_i(x) \leq 1, \quad i \in I,$$

*and/or a given finite collection of linear equations*

$$g_j(x) = 1, \quad j \in J,$$

*where each $f_i$ or $g_j$ is a linear functional on $V$. Then for any $x \in C$, we have*

$$\Phi(x : C) = \{y \in \Phi(x : K) \textstyle\bigcap C : \ \forall i \in I,\ f_i(x) = 1 \textit{ implies } f_i(y) = 1\}.$$

*In particular, if $x \in C$ is such that $f_i(x) < 1$ for all $i \in I$, then $\Phi(x : C) = \Phi(x : K) \bigcap C$.*

*Proof.* Let $y \in \Phi(x : C)$. Then clearly $y \in C$, and for some $\mu > 0$, we have $x+\mu(x-y) \in C \subseteq K$, hence $y \in \Phi(x : K) \bigcap C$. Let $i \in I$ be such that $f_i(x) = 1$. Then $1 \geq f_i(x + \mu(x - y)) = 1+\mu f_i(x-y)$, and hence $f_i(x) \leq f_i(y)$. But $f_i(x) = 1$ and $f_i(y) \leq 1$, it follows that $f_i(y) = 1$. This proves one of the inclusions.

Conversely, let $y \in \Phi(x : K) \bigcap C$ be such that for any $i \in I$, $f_i(y) = 1$ whenever $f_i(x) = 1$. Since $y \in \Phi(x : K)$, $x + \mu(x - y) \in K$ for any $\mu > 0$ sufficiently small. Clearly, for any $j \in J$, we have $g_j(x + \mu(x - y)) = 1$, as $g_j(x) = g_j(y) = 1$. By our assumption on $y$, it is also clear that for any $i \in I$ for which $f_i(x) = 1$, we have $f_i(x + \mu(x - y)) = 1$. On the other hand, if $i \in I$ is such that $f_i(x) < 1$, then clearly we have $f_i(x + \mu(x - y)) < 1$ for $\mu > 0$ sufficiently small. This shows that for $\mu > 0$ sufficiently small, we have $x + \mu(x - y) \in C$, thus establishing the reverse inclusion. □

In our applications, we often take (i) $K$ to be the cone of $m \times n$ nonnegative matrices, (ii) $f_i(X)$ to be a certain entry of $X$, or a certain row sum or column sum of $X$, and (iii) $g_i(X)$ to be a certain row sum or column sum of $X$. Note that when $K$ is the cone of $m \times n$ nonnegative matrices and $X$ is an element in $K$, a matrix $Y \in K$ satisfies $Y \in \Phi(X : K)$ if and only if the $(i, j)$ entry of $Y$ is zero whenever the $(i, j)$ entry of $X$ is zero.

## 2 Permutation and doubly stochastic matrices

In this section, we characterize those linear maps $T$ on $\text{span}(\mathcal{S})$ satisfying $T(\mathcal{S}) = \mathcal{S}$ for $\mathcal{S} = \mathbf{P}(n)$ or $\mathbf{DS}(n)$. By the Birkhoff Theorem (e.g., see [3]), we see that $\mathcal{E}(\mathbf{DS}(n)) = \mathbf{P}(n)$. One easily checks that $\text{span}(\mathbf{DS}(n)) = \text{span}(\mathbf{P}(n))$ is the subspace of $n \times n$ real matrices with equal row sums and column sums, and has dimension $(n-1)^2 + 1$. By Proposition 1.2, we have the following.

**Lemma 2.1** *Let $A = (a_{ij}) \in \mathbf{DS}(n)$. Then $B = (b_{ij}) \in \mathbf{P}(n)$ satisfies $B \in \Phi(A : \mathbf{DS}(n))$ if any only if $b_{ij} = 0$ whenever $a_{ij} = 0$.*

**Theorem 2.2** *Let $T$ be a linear map on* $\text{span}(\mathbf{DS}(n))$. *The following conditions are equivalent:*

(a) $T(\mathbf{DS}(n)) = \mathbf{DS}(n)$.

(b) $T(\mathbf{P}(n)) = \mathbf{P}(n)$.

(c) *$T$ is of the form $X \mapsto PXQ$* or *$X \mapsto PX^tQ$ for some $P, Q \in \mathbf{P}(n)$.*

*Proof.* The equivalence of (a) and (b) follows from the fact that $\mathcal{E}(\mathbf{DS}(n)) = \mathbf{P}(n)$, and the implication (c) $\Rightarrow$ (b) is clear. It remains to show that if (a) and (b) hold then so does (c). We assume $n \geq 3$ (as the case of $n \leq 2$ is easy to verify), and divide the proof into several steps. Our strategy is to show that if (a) and (b) hold, then there exist $P_1, \ldots, P_r, Q_1, \ldots, Q_s \in \mathbf{P}(n)$ such that $\widetilde{T}$ defined by $\widetilde{T}(X) = P_r \cdots P_1 T(X) Q_1 \cdots Q_s$ or $\widetilde{T}(X) = P_r \cdots P_1 T(X^t) Q_1 \cdots Q_s$ is the identity map. It will then follow that $T$ satisfies condition (c).

Hereafter, we always assume that $n \geq 3$ and $T$ satisfies (a) and (b). Furthermore, we may assume that $T(I_n) = I_n$; otherwise we may replace $T$ by $\widetilde{T}$ defined by $\widetilde{T}(X) = T(I_n)^t T(X)$, noting that $T(I_n) \in \mathbf{P}(n)$.

First of all, we have the following simple observation.

**Assertion 1** $T(J_n) = J_n$.

*Proof of Assertion 1.* Since $\sum_{P \in \mathbf{P}(n)} P = (n-1)! J_n$, the assertion follows from

$$(n-1)!\, T(J_n) = \sum_{P \in \mathbf{P}(n)} T(P) = \sum_{P \in \mathbf{P}(n)} P = (n-1)! J_n.$$

Let $\mathcal{C}_2$ denote the set of all $n \times n$ permutation matrices corresponding to transpositions, i.e., elements of the form $P(i,j)$. We have the following.

**Assertion 2** *Suppose $T(I_n) = I_n$. Then $T(\mathcal{C}_2) = \mathcal{C}_2$.*

*Proof of Assertion 2.* Note that $P_1 \in \mathcal{C}_2$ if and only if $P_1 \neq I_n$ and there exist distinct elements $P_j \in \mathbf{P}(n)$, all different from $I_n$, for $2 \leq j \leq k = n(n-1)/2$ such that

$$P_1 + \cdots + P_k = (J_n - I_n) + \frac{(n-1)(n-2)}{2} I_n.$$

(To show the "if" part, we equate the sum of all off-diagonal entries of the matrices on the two sides of the preceding equation. Then we readily see that each $P_j$ has precisely two nonzero off-diagonal entries and hence must belong to $\mathcal{C}_2$.) Thus, if $P_1 \in \mathcal{C}_2$, then $T(P_1) \in \mathcal{C}_2$, i.e., $T(\mathcal{C}_2) \subseteq \mathcal{C}_2$. Since $T$ is injective, the assertion follows.

**Assertion 3** *Suppose $T(I_n) = I_n$. Let $P(i,j), P(k,l) \in \mathcal{C}_2$, and $P(p,q) = T(P(i,j)), P(r,s)$ $= T(P(k,l))$. If $\{i,j\} \cap \{k,l\}$ is a singleton set, then so is $\{p,q\} \cap \{r,s\}$.*

*Proof of Assertion 3.* Without lost of generality let $i = k$, and $i, j, l$ be distinct. As $P(i,j)$ and $P(i,l)$ are distinct, their image under the injective map $T$ must be distinct. Hence $\{p,q\} \cap \{r,s\}$ cannot have two elements. On the other hand, if $\{p,q\} \cap \{r,s\} = \emptyset$ then write $A = (P(i,j) + P(i,l))/2$. By Lemma 2.1, $\Phi(A : \mathbf{DS}(n)) \cap \mathcal{E}(\mathbf{DS}(n))$ has two elements (namely, $P(i,j)$ and $P(i,l)$), whereas $\Phi(T(A) : \mathbf{DS}(n)) \cap \mathcal{E}(\mathbf{DS}(n)$ has four (namely, $P(p,q), P(r,s), P(p,q)P(r,s)$ and $I_n$). This contradicts Proposition 1.1. Hence Assertion 3 holds.

Next, we may further assume that $T(P(1,2)) = P(1,2)$; otherwise replace $T$ by $\widetilde{T}$ defined by $\widetilde{T}(X) = P_\sigma^t T(X) P_\sigma$, where $\sigma \in S_n$ satisfies $T(P(1,2)) = P(\sigma(1), \sigma(2))$.

**Assertion 4** *Suppose $T$ fixes $I_n$ and $P(1,2)$. Then there exists $P \in \mathbf{P}(n)$ such that $T(X) = P^t X P$ for all $X \in \mathcal{C}_2$.*

*Proof of Assertion 4.* Since $T$ fixes $P(1,2)$, by Assertion 3 (with $(i,j) = (1,2)$ and $(k,l) = (1,3)$), $T$ maps $P(1,3)$ to either $P(1,s)$ or $P(2,s)$ for some $s \geq 3$. We may assume $T(P(1,3)) = P(1,s)$; otherwise replace $T$ by $\widetilde{T}$ defined by $\widetilde{T}(X) = P(1,2)T(X)P(1,2)$. Further, we may assume $T(P(1,3)) = P(1,3)$; otherwise replace $T$ by $\widetilde{T}$ defined by $\widetilde{T}(X) = P(3,s)T(X)P(3,s)$.

Suppose $l \geq 4$. Since $T$ fixes $P(1,2)$ and $P(1,3)$, by Assertion 3 again (with $(i,j) = (1,2)$ and $(1,3)$ in turn, and $(k,l) = (1,l)$), we deduce that $T(P(1,l))$ must be of the form $P(1,s_l)$ for some $s_l \geq 4$ or equal to $P(2,3)$. But, in view of Proposition 1.1 and Lemma 2.1, the latter case cannot happen because $\Phi\left(\frac{P(1,2)+P(1,3)+P(1,l)}{3} : \mathbf{DS}(n)\right) \cap \mathcal{E}(\mathbf{DS}(n))$ contains 3 elements while $\Phi\left(\frac{P(1,2)+P(1,3)+P(2,3)}{3} : \mathbf{DS}(n)\right) \cap \mathcal{E}(\mathbf{DS}(n))$ contains 6. We may further assume that $s_l = l$ for all $l \geq 4$, so that $T$ fixes all $P(1,i)$ for all $i \geq 2$; otherwise replace $T$ by $\widetilde{T}$ defined by $\widetilde{T}(X) = P_\sigma^t T(X) P_\sigma$ where $\sigma \in S_n$ is given by $\sigma(i)$ equals $i$ for $i = 1,2,3$ and equals $s_i$ for $i = 4, \ldots, n$.

Now consider any distinct $u, v \in \{2, \ldots, n\}$. As $T$ fixes $P(1,u)$ and $P(1,v)$, by Assertion 3 (with $(i,j) = (1,u)$ and $(1,v)$ in turn, and $(k,l) = (u,v)$) we conclude that $T$ fixes $P(u,v)$ also. Hence Assertion 4 holds.

We may further assume that $P = I_n$ in Assertion 4; otherwise replace $T$ by $\widetilde{T}$ defined by $\widetilde{T}(X) = PT(X)P^t$.

**Assertion 5** *Suppose $T(X) = X$ for $X = I_n$ or $X \in \mathcal{C}_2$. Let $Y \in \mathbf{P}(n)$ have the form $P(i_1, \ldots, i_k)$ with $k \geq 3$. Then $\{T(Y), T(Y^t)\} = \{Y, Y^t\}$. Furthermore, if $T(P(1,2,3)) = P(1,2,3)$, then $T(Y) = Y$.*

*Proof of Assertion 5.* Note that $Y + Y^t = P(i_1, \ldots, i_k) + P(i_1, \ldots, i_k)^t = P(i_1, i_2) + P(i_2, i_3) + \cdots + P(i_k, i_1) - (k-2)I_n$. As $T$ fixes $I_n$ and all elements in $\mathcal{C}_2$, it fixes $Y + Y^t$ also. Note that it is possible to write $T(Y) + T(Y^t)$ $(= Y + Y^t)$ as a sum $Z_1 + Z_2$ for some $Z_1, Z_2 \in \mathbf{P}(n)$ with $\{Z_1, Z_2\} \neq \{Y, Y^t\}$ only when $k = 2m$ is even and

$$\{Z_1, Z_2\} = \{P(i_1, i_2)P(i_3, i_4)\cdots P(i_{2m-1}, i_{2m}),\ P(i_2, i_3)P(i_4, i_5)\cdots P(i_{2m}, i_1)\}.$$

Since $P(i_1, i_2)P(i_3, i_4)\cdots P(i_{2m-1}, i_{2m}) = P(i_1, i_2) + P(i_3, i_4) + \cdots + P(i_{2m-1}, i_{2m}) - (m-1)I_n$, it is fixed by $T$, and therefore $T$ must fix $Z_1$ and $Z_2$. Consequently, we cannot have $\{T(Y), T(Y^t)\} = \{Z_1, Z_2\}$. By our assumption on $T$ and $Y$, we must have $\{T(Y), T(Y^t)\}$ equals $\{Y, Y^t\}$.

Next we observe that if $P(j_1, j_2, j_3)$ and $Y$ have at least one common nonzero off-diagonal position, and if $T$ fixes $P(j_1, j_2, j_3)$, then $T$ fixes $Y$ also. To see this, assume without loss of generality that $j_1 = i_1$ and $j_2 = i_2$. By Lemma 2.1, $\Phi\left(\frac{I_n + P(j_1,j_2,j_3) + Y^t}{3} : \mathbf{DS}(n)\right) \cap \mathcal{E}(\mathbf{DS}(n))$ contains an element of $\mathcal{C}_2$ (namely $P(i_1, i_2)$), whereas $\Phi\left(\frac{I_n + P(j_1,j_2,j_3) + Y}{3} : \mathbf{DS}(n)\right) \cap \mathcal{E}(\mathbf{DS}(n))$ does not. Since $T$ fixes every element in $\mathcal{C}_2$, the above observation and Proposition 1.1 imply that $T(Y) \neq Y^t$, and hence $T$ must fix $Y$.

Now we assume in addition that $T$ fixes $P(1,2,3)$. Then $T$ fixes its transpose $P(1,3,2)$ as well, because $\{T(X), T(X^t)\} = \{X, X^t\}$ for $X = P(1,2,3)$. As the nonzero off-diagonal positions $(i,j)$ of $P(1,2,3)$ and $P(1,3,2)$ cover all possible $(i,j)$ for distinct $i, j \in \{1,2,3\}$, by the observation in the above paragraph, $T$ fixes all $P(i,j,p)$ where $i, j, p$ are distinct and $i, j \in \{1,2,3\}$. Using the same argument, we deduce that $T$ fixes all $P(j,p,q)$ where $j, p, q$ are distinct and $j \in \{1,2,3\}$. Note that the ordered pairs $(p,q)$ obtained from such $p, q$'s cover all possible off-diagonal positions. Using the same argument again, we conclude that $T$ must fix any $P(i_1, \ldots, i_k)$ with $k \geq 3$. This completes the proof of Assertion 5.

In view of Assertion 5, when $T(X) = X$ for $X = I_n$ or $X \in \mathcal{C}_2$, replacing $T$ by $\widetilde{T}$ defined by $\widetilde{T}(X) = T(X^t)$ if necessary, we may assume that $T(X) = X$ whenever $X$ is of the form $P(i_1, \ldots, i_k)$ with $k \geq 2$. We complete the proof by establishing the following assertion.

**Assertion 6** *Suppose $T(X) = X$ whenever $X = I_n$ or $X$ is of the form $P(i_1, \ldots, i_k)$, where $k \geq 2$. Then $T(X) = X$ for all $X \in \mathbf{P}(n)$.*

*Proof of Assertion 6.* Consider any $P \in \mathbf{P}(n)$. Let $P = P_1 \cdots P_m$ be a disjoint cycle decomposition for $P$. Observe that $P_1 \cdots P_m = P_1 + \cdots + P_m - (m-1)I_n$. Since $T$ fixes $I_n$ and each $P_j$, it fixes $P$ also. □

## 3 Subpermutation and substochastic matrices

In this section, we characterize linear maps $T$ satisfying $T(\mathcal{S}) = \mathcal{S}$ where $\mathcal{S}$ is any of $\mathbf{DsS}(m,n), \mathbf{sP}(m,n), \mathbf{CS}(m,n), \mathbf{CsS}(m,n), \mathbf{RS}(m,n)$, and $\mathbf{RsS}(m,n)$. We consider the case of $\mathcal{S} = \mathbf{DsS}(m,n)$ or $\mathbf{sP}(m,n)$ first.

It is known that $\mathcal{E}(\mathbf{DsS}(m,n)) = \mathbf{sP}(m,n)$. One can use Proposition 1.2 and the fact that $\mathbf{sP}(m,n)$ contains all $E_{ij}$ to prove the next lemma, and conclude that $\text{span}(\mathbf{DsS}(m,n)) = \text{span}(\mathbf{sP}(m,n)) = \mathbb{R}^{m\times n}$.

**Lemma 3.1** *Let $\mathcal{S} = \mathbf{DsS}(m,n)$. Then $\mathcal{E}(\mathcal{S}) = \mathbf{sP}(m,n)$. Suppose $A = (a_{ij}) \in \mathcal{S}$ and $B = (b_{ij}) \in \mathcal{E}(\mathcal{S})$. Then $B \in \Phi(A : \mathcal{S})$ if any only if $b_{ij} = 0$ whenever $a_{ij} = 0$, and whenever a row (column, respectively) sum of $A$ is 1 then so is the corresponding row (column, respectively) sum of $B$.*

**Theorem 3.2** *Let $T$ be a linear map on $\mathbb{R}^{m\times n}$. The following conditions are equivalent:*

(a) $T(\mathbf{DsS}(m,n)) = \mathbf{DsS}(m,n)$.

(b) $T(\mathbf{sP}(m,n)) = \mathbf{sP}(m,n)$.

(c) *There exist $P \in \mathbf{P}(m)$ and $Q \in \mathbf{P}(n)$ such that*

  (i) *$T$ is of the form $X \mapsto PXQ$, or*

  (ii) *$m = n$ and $T$ is of the form $X \mapsto PX^tQ$.*

*Proof.* As in the proof of Theorem 2.2, we need only to show that if (a) and (b) hold then so does (c).

First observe that, for any $A \in \mathbf{sP}(m,n)$, by Lemma 3.1, $A$ has exactly one nonzero entry (and hence is in the form of $E_{ij}$) if and only if $\Phi(A/2 : \mathbf{DsS}(m,n)) \cap \mathcal{E}(\mathbf{DsS}(m,n))$ contains exactly 2 elements (namely, 0 and $A$). By (b) and Proposition 1.1, we have

**Assertion 1** *For all $E_{i,j} \in \mathbb{R}^{m\times n}$, it holds that $T(E_{ij}) = E_{pq}$ for some $p, q$.*

Next observe that, for any distinct $E_{i_1j_1}, E_{i_2j_2} \in \mathbb{R}^{m\times n}$, their sum is not in $\mathbf{sP}(m,n)$ if and only if the two nonzero entries of $E_{i_1j_1}$ and $E_{i_2j_2}$ are either in a same row or in a same column. By (b) and Assertion 1, and by considering $T(E_{i_1j_1} + E_{i_2j_2})$, we have

**Assertion 2** *For any distinct $E_{i_1j_1}, E_{i_2j_2} \in \mathbb{R}^{m\times n}$, let $T(E_{i_1j_1}) = E_{p_1q_1}$, $T(E_{i_2j_2}) = E_{p_2q_2}$. If the two nonzero entries of $E_{i_1j_1}$ and $E_{i_2j_2}$ are in either a same row or a same column, then so are the two nonzero entries of $E_{p_1q_1}$ and $E_{p_2q_2}$.*

Suppose $T(E_{11}) = E_{pq}$. Then by Assertions 1 and 2, for all $j = 2, \ldots, n$, the nonzero entry of each $T(E_{1j})$ must be on either the $p$th row or the $q$th column. Note that one cannot have $T(E_{1j_1}) = E_{pq'}$ and $T(E_{1j_2}) = E_{p'q}$ for some distinct $j_1, j_2 \geq 2$ and $p' \neq p$, $q' \neq q$;

otherwise the nonzero entries of $E_{pq'}$ and $E_{p'q}$ are not in the same row (column), contradicting Assertion 2. Hence $T$ maps $E_{1*}$ either onto $E_{p*}$ or into $E_{*q}$, where $E_{i*}$ and $E_{*j}$ denote the sets $\{E_{ij} \in \mathbb{R}^{m\times n} : 1 \le j \le n\}$ and $\{E_{ij} \in \mathbb{R}^{m\times n} : 1 \le i \le m\}$ respectively. By considering the image of $E_{*1}$ under $T$ and the image of $E_{p*}$ and $E_{*q}$ under $T^{-1}$, we conclude that either (i) $T(E_{1*}) = E_{p*}$ and $T(E_{*1}) = E_{*q}$, or (ii) $m = n$, and $T(E_{1*}) = E_{*q}$ and $T(E_{*1}) = E_{p*}$.

Suppose (i) holds. Then there exist permutations $\sigma \in S_m$, $\tau \in S_n$ such that $\sigma(1) = p, \tau(1) = q$, and $T(E_{ij}) = E_{\sigma(i)\tau(j)}$ whenever $i = 1$ or $j = 1$. Now consider any $E_{ij}$ with $i, j \ge 2$. By Assertion 2 (with $(i_1, j_1) = (i, j)$, and $(i_2, j_2) = (1, j)$ and $(i, 1)$ in turn), and noting that $T(E_{ij}) \ne T(E_{11}) = E_{pq}$, we conclude that $T(E_{ij}) = E_{\sigma(i)\tau(j)}$. Hence $T$ is of the desired form in (c)(i).

If (ii) holds, then by replacing $T$ by the linear map of the form $X \mapsto (T(X))^t$, we are back to case (i). Consequently, $T$ is of the form (c)(ii). □

Next we consider the case of $\mathcal{S} = \mathbf{CS}(m, n)$. Again, one can use Proposition 1.2 to prove the next lemma and conclude that $\mathrm{span}(\mathbf{CS}(m, n))$ is the subspace of all $m \times n$ real matrices with all column sums equal.

**Lemma 3.3** *Let $\mathcal{S} = \mathbf{CS}(m, n)$. Then*

$$\mathcal{E}(\mathcal{S}) = \{A \in \mathbb{R}^{m\times n} : \textit{each column of } A \textit{ is a coordinate vector in } \mathbb{R}^m\}.$$

*Suppose $A = (a_{ij}) \in \mathcal{S}$ and $B = (b_{ij}) \in \mathcal{E}(\mathcal{S})$. Then $B \in \Phi(A : \mathcal{S})$ if any only if $b_{ij} = 0$ whenever $a_{ij} = 0$.*

**Theorem 3.4** *Let $T$ be a linear map on* $\mathrm{span}(\mathbf{CS}(m, n))$. *The following conditions are equivalent:*

(a) *$T(\mathbf{CS}(m, n)) = \mathbf{CS}(m, n)$.*

(b) *$T$ maps the set $\{A \in \mathbb{R}^{m\times n}$ : each column of $A$ is a coordinate vector in $\mathbb{R}^m\}$ onto itself.*

(c) *There exist $P_1, \ldots, P_n \in \mathbf{P}(m)$ and $Q \in \mathbf{P}(n)$ such that $T$ has the form*

$$[A_1|\cdots|A_n] \mapsto [P_1A_1|\cdots|P_nA_n]\, Q.$$

*Proof.* As in the proof of Theorem 2.2, we need only to show that if (a) and (b) hold then so does (c).

Let $e_1, \ldots, e_m$ denote the coordinate vectors in $\mathbb{R}^m$, and let $Z = [e_1|\cdots|e_1] \in \mathbb{R}^{m\times n}$. Note that $Z \in \mathcal{E}(\mathbf{CS}(m, n))$. By (b), we may assume without loss of generality that $T(Z) = Z$; otherwise $T(Z) = [e_{i_1}|\cdots|e_{i_n}]$ for some $i_1, \ldots, i_n \in \{1, \ldots, m\}$, and we may replace $T$

by the composite map $\widetilde{T} \circ T$ where $\widetilde{T}$ has the form $[A_1|\cdots|A_n] \mapsto [P_1A_1|\cdots|P_nA_n]$ with $P_1,\ldots,P_n \in \mathbf{P}(n)$ satisfying $P_je_{i_j} = e_1$ for all $j$. We shall need the following in our proof:

**Assertion 1** *Suppose $X, Y \in \mathcal{E}(\mathbf{CS}(m,n))$. If $X$ and $Y$ are such that the 1-entries of $X$ differ from those of $Y$ by $k$ positions, then so are $T(X)$ and $T(Y)$.*

*Proof of Assertion 1.* Note that, by Lemma 3.3, the 1-entries of $X$ differ from those of $Y$ by $k$ positions if and only if $\Phi\left(\frac{X+Y}{2} : \mathbf{CS}(m,n)\right) \cap \mathcal{E}(\mathbf{CS}(m,n))$ contains exactly $2^k$ elements. By (b) and Proposition 1.1, the assertion holds.

Let $i = 2,\ldots,m$, and let $Y_i \in \mathcal{E}(\mathbf{CS}(m,n))$ be such that its first column is $e_i$ and the remaining columns are $e_1$. Since $T$ fixes $Z$, and the 1-entries of $Y_i$ differ from those of $Z$ by one position, it follows from Assertion 1 that exactly one column of $Y_i$ is a coordinate vector not equal to $e_1$. Moreover, for any distinct $i_1, i_2 \in \{2,\ldots,m\}$, it cannot happen that the two 1-entries of $T(Y_{i_1})$ and $T(Y_{i_2})$ not on the first row are on different columns; otherwise $Y_{i_1}$ and $Y_{i_2}$ have $n-1$ common 1-entry positions whereas $T(Y_{i_1})$ and $T(Y_{i_2})$ have $n-2$, which contradicts Assertion 1. Accordingly, $T(Y_{*1}) = Y_{*q_1}$ for some $q_1 \in \{1,\ldots,n\}$, where $Y_{*j}$ denotes the set of matrices in $\mathcal{E}(\mathbf{CS}(m,n))$ the $j$th column of which is $e_i$ $(i = 1,\ldots,m)$ and the remaining columns are $e_1$. Similarly, we have $T(Y_{*j}) = Y_{*q_j}$ for all $j = 2,\ldots,n$, where $q_j \in \{1,\ldots,n\}$. As $T$ is injective, there exist $P_1,\ldots,P_n \in \mathbf{P}(m)$ and $Q \in \mathbf{P}(n)$ such that $P_je_1 = e_1$ for all $j = 1,\ldots,n$ and $T([e_{i_1}|\cdots|e_{i_n}]) = [P_1e_{i_1}|\cdots|P_ne_{i_n}]\,Q$ whenever one or less of $i_1,\ldots,i_n$ differ from 1. We may assume without loss of generality that $P_1 = \cdots = P_n = I_m$ and $Q = I_n$; otherwise replace $T$ by the composite map $\widetilde{T}^{-1} \circ T$ where $\widetilde{T}$ maps all $[A_1|\cdots|A_n] \in \operatorname{span}(\mathbf{CS}(m,n))$ to $[P_1A_1|\cdots|P_nA_n]\,Q$. To summarize, we conclude that the following holds for $k = 1$:

**Assertion 2** *$T$ fixes $[e_{i_1}|\cdots|e_{i_n}]$ whenever $k$ or less of $i_1,\ldots,i_n$ differ from 1.*

Now suppose Assertion 2 holds for some $k$ with $1 \le k < n$, and let $Y = [e_{i_1}|\cdots|e_{i_n}]$ be such that $k+1$ of $i_1,\ldots,i_n$ differ from 1. Without loss of generality assume $i_1 = \cdots = i_{k+1} = 2$ and $i_{k+2} = \cdots = i_n = 1$. For $j = 1, 2$, let $X_j$ be obtained from $Y$ by changing the $j$th column of $Y$ to $e_1$. As $T$ fixes $X_1, X_2$ by assumption, and the 1-entries of $Y$ differ from those of each of $X_1, X_2$ by one position, it follows from Assertion 1 that $T(Y)$ equals either $Y$ or $X_0$, where $X_0$ is obtained from $Y$ by changing the first two columns of $Y$ to $e_1$. However, the latter case cannot happen because $T$ fixes $X_0$ by assumption and $T$ is injective. Hence $T$ fixes $Y$. By induction, we see that $T$ fixes all elements of $\mathcal{E}(\mathbf{CS}(m,n))$, and (c) follows. □

Now we consider the case $\mathcal{S} = \mathbf{CsS}(m,n)$. Again, one can use Proposition 1.2 to prove the next lemma and then conclude that $\operatorname{span}(\mathbf{CsS}(m,n))$ is $\mathbb{R}^{m\times n}$.

**Lemma 3.5** *Let $\mathcal{S} = \mathbf{CsS}(m,n)$. Then*

$$\mathcal{E}(\mathcal{S}) = \{A \in \mathbb{R}^{m\times n} : \text{each column of } A \text{ is either zero or a coordinate vector in } \mathbb{R}^m\}.$$

*Suppose $A = (a_{ij}) \in \mathcal{S}$ and $B = (b_{ij}) \in \mathcal{E}(\mathcal{S})$. Then $B \in \Phi(A : \mathcal{S})$ if any only if $b_{ij} = 0$ whenever $a_{ij} = 0$, and whenever a column sum of $A$ is 1 then so is the corresponding column sum of $B$.*

**Theorem 3.6** *Let $T$ be a linear map on $\mathbb{R}^{m\times n}$. The following conditions are equivalent:*

(a) $T(\mathbf{CsS}(m,n)) = \mathbf{CsS}(m,n)$.

(b) *$T$ maps the set $\{A \in \mathbb{R}^{m\times n}$ : each column of $A$ is either zero or a coordinate vector in $\mathbb{R}^m\}$ onto itself.*

(c) *There exist $P_1, \ldots, P_n \in \mathbf{P}(m)$ and $Q \in \mathbf{P}(n)$ such that $T$ has the form*

$$[A_1|\cdots|A_n] \mapsto [P_1A_1|\cdots|P_nA_n]\,Q.$$

*Proof.* Similar to the proof of Theorem 2.2, we need only to show that if (a) and (b) hold then so does (c). First observe that, by Lemma 3.5, $A \in \mathcal{E}(\mathbf{CsS}(m,n))$ is equal to $E_{ij}$ for some $i, j$ if and only if $\Phi(\frac{A}{2} : \mathbf{CsS}(m,n)) \cap \mathcal{E}(\mathbf{CsS}(m,n))$ contains exactly two elements (namely, 0 and $A$). By (b) and Proposition 1.1, we have $T(E_{ij}) = E_{pq}$ for some $p, q$.

Next observe that for any $E_{i_1j_1}, E_{i_1j_2} \in \mathbb{R}^{m\times n}$, their sum is not contained in $\mathcal{E}(\mathbf{CsS}(m,n))$ if and only if $j_1 = j_2$. As $T$ strongly preserves $\mathcal{E}(\mathbf{CsS}(m,m))$, we deduce that $T(E_{*j}) = E_{*q_j}$ for some $q_j \in \{1,\ldots,n\}$, where $E_{*j}$ denotes the set $\{E_{ij} \in \mathbb{R}^{m\times n} : 1 \le i \le m\}$. Then it follows that there exist permutations $\sigma_1, \ldots, \sigma_n \in S_m$, $\tau \in S_n$ such that $T(E_{ij}) = E_{\sigma_j(i),\tau(j)}$ for all $E_{ij} \in \mathbb{R}^{m\times n}$. Therefore $T$ is of the desired form in (c). □

It should be clear that results on $\mathbf{RS}(m,n)$ and $\mathbf{RsS}(m,n)$ parallel those of Lemmas 3.3, 3.5 and Theorems 3.4, 3.6 can be readily obtained by changing $\mathbf{CS}(m,n)$, $\mathbf{CsS}(m,n)$ and the words "column" to $\mathbf{RS}(m,n), \mathbf{RsS}(m,n)$ and "row" respectively.

# 4 Linear preservers of $\mathcal{S}$

In this section we consider linear maps $T$ that satisfy $T(\mathcal{S}) \subseteq \mathcal{S}$, where $\mathcal{S}$ is one of the sets treated in the previous sections. For some particular subsets $C$, one may be able to show that $T(C) \subseteq C$ automatically implies $T(C) = C$ (see [5]), and hence the problem of characterizing linear preservers of $C$ reduces to that of characterizing strong preservers. But in other situations, linear preservers which are not strong preservers do exist, and the problem of characterizing them is usually more difficult or even intractable. Our study falls into the latter category.

If $T(\mathbf{P}(n)) = R \subseteq \mathbf{P}(n)$, then $T(\mathbf{DS}(n))$ is the convex hull of $R$. Also, if $T(\mathbf{sP}(m,n)) = S \subseteq \mathbf{sP}(m,n)$, then $T(\mathbf{DsS}(m,n))$ is the convex hull of $S$. So, one can study linear preservers of $\mathbf{P}(n)$ and $\mathbf{sP}(m,n)$ via the linear preservers of $\mathbf{DS}(n)$ and $\mathbf{DsS}(m,n)$ and focus on the linear preservers of $\mathcal{S} = \mathbf{DS}(n), \mathbf{DsS}(m,n), \mathbf{CS}(m,n)$, and $\mathbf{CsS}(m,n)$. (The results on $\mathbf{RS}(m,n)$ and $\mathbf{RsS}(m,n)$ are parallel to those of $\mathbf{CS}(m,n)$ and $\mathbf{CsS}(m,n)$.)

If $\mathcal{S}$ is a convex polytope, then the set $\pi(\mathcal{S})$ of linear maps on $\text{span}(\mathcal{S})$ satisfying $T(\mathcal{S}) \subseteq \mathcal{S}$ is also a polytope; e.g., see Theorem 7 of [7]. Since every element in $\pi(\mathcal{S})$ is a convex combination of the extreme elements of $\pi(\mathcal{S})$, it is interesting to study the set of extreme elements of $\pi(\mathcal{S})$.

We have the following observation concerning extreme elements of $\pi(C)$, where $C$ is a general compact convex set.

**Proposition 4.1** *Let $C$ be a compact convex set. If $T \in \pi(C)$ is such that $T(x) \in \mathcal{E}(C)$ for all (or $\dim(\text{span}(C))$ many linearly independent) $x \in \mathcal{E}(C)$, then $T$ is an extreme element of $\pi(C)$.*

*Proof.* Let $R, S \in \pi(C)$ and $0 < \alpha < 1$ be such that $\alpha R + (1-\alpha)S = T$. For any $x \in \mathcal{E}(C)$, we have $\alpha R(x) + (1-\alpha)S(x) = T(x)$. If $T(x) \in \mathcal{E}(C)$ then, since $R(x), S(x) \in C$, we have $R(x) = S(x) = T(x)$. As the set of such $x$'s is a spanning set of $\text{span}(C)$, it follows that $R = S = T$. □

It follows from the above proposition that the identity map is always an extreme element of $\pi(C)$. Denote by $\text{aff}\, C$ the affine hull of $C$. We have the following corollary.

**Corollary 4.2** *Let $C$ be a simplex. Suppose $0 \notin \text{aff}\, C$. Then $T \in \pi(C)$ is an extreme element if and only if $T(\mathcal{E}(C)) \subseteq \mathcal{E}(C)$.*

*Proof.* The "if" part follows from Proposition 4.1. To prove the "only if" part, let $\mathcal{E}(C) = \{x_1, \cdots, x_n\}$, which is a basis for $\text{span}(C)$ because $C$ is a simplex and $0 \notin \text{aff}\, C$. Let $T \in \pi(C)$ and suppose that $T(\mathcal{E}(C)) \not\subset \mathcal{E}(C)$, say, $T(x_1) = (u+v)/2$ where $u, v \in C$, $u \neq v$. Define $R$ and $S$ respectively by $R(x_1) = u$, $S(x_1) = v$, $R(x_i) = S(x_i) = T(x_i)$ for $i = 2, \ldots, n$. Clearly, we have $R, S \in \pi(C)$, $R \neq S$, and $T = (R+S)/2$. Thus $T$ is not an extreme element in $\pi(C)$. □

If $C$ is a compact convex set for which $0 \notin \text{aff}\, C$, then the rank-one elements of $\pi(C)$ are simply all the linear maps which take $C$ to a fixed element $p$ of $C$. It is clear that such a map is an extreme element of $\pi(C)$ if and only if the corresponding element $p$ is an extreme element of $C$. Since $0 \notin \text{aff}\, C$, there exists a linear functional $g$ on $\text{span}(C)$ such that $g(C) = \{1\}$. So the rank-one elements of $\pi(C)$ can also be expressed as $x \mapsto g(x)p$, where $p \in C$. More generally, if $f_1, \ldots, f_k$ are nonzero linear functionals on $\text{span}(C)$, each nonnegative on $C$, such that $f_1 + \cdots + f_k$ takes the constant value 1 on $C$, then the linear map on $\text{span}(C)$ given by $x \mapsto \sum_{i=1}^k f_i(x)x_i$, where $x_1, \ldots, x_k$ are fixed elements of $C$, belongs to $\pi(C)$.

Applying the results in Sections 2 and 3 and the above discussion to our sets $\mathcal{S}$, we obtain the following.

**Proposition 4.3** *Let* $\mathcal{S} = \mathbf{DS}(m), \mathbf{DsS}(m,n), \mathbf{CS}(m,n)$*, or* $\mathbf{CsS}(m,n)$*, and let* $T$ *be a linear map on* $\mathrm{span}(\mathcal{S})$*. Then* $T$ *is an extreme element of* $\pi(\mathcal{S})$ *if one of the following holds.*

(a) $T$ *satisfies* $T(\mathcal{S}) = \mathcal{S}$.

(b) $T$ *is of the form* $(x_{ij}) \mapsto (\sum_{i=1}^m x_{ij})\, P$ *for some* $j = 1, \dots, n$*, where* $P \in \mathcal{E}(\mathcal{S})$.

(c) $T$ *is of the form* $(x_{ij}) \mapsto \sum_{i=1}^m x_{ij} R_i$ *for some* $j = 1, \dots, n$*, where* $R_1, \dots, R_m \in \mathcal{E}(\mathcal{S})$.

(d) $\mathcal{S} = \mathbf{DS}(n)$ *or* $\mathbf{DsS}(m,n)$*, and* $T$ *is of the form* $(x_{ij}) \mapsto \sum_{j=1}^n x_{ij} R_j$ *for some* $i = 1, \dots, m$*, where* $R_1, \dots, R_n \in \mathcal{E}(\mathcal{S})$.

(e) $\mathcal{S} = \mathbf{DsS}(m,n)$ *and there exist* $P \in \mathbf{sP}(m)$ *and* $Q \in \mathbf{sP}(n)$ *such that*

(i) $T$ *is of the form* $PAQ$*, or*

(ii) $m = n$ *and* $T$ *is of the form* $PA^tQ$.

(f) $\mathcal{S} = \mathbf{CsS}(m,n)$ *and there exist* $P_1, \dots, P_n \in \mathbf{sP}(m)$ *and* $Q \in \mathbf{sP}(n)$ *such that* $T$ *is of the form* $[A_1|\cdots|A_n] \mapsto [P_1A_1|\cdots P_nA_n]\, Q$.

As mentioned before, finding all the extreme elements of $\pi(\mathcal{S})$ is an intricate problem. We illustrate this by considering $\mathcal{S} = \mathbf{DS}(n)$. Since $\mathbf{DS}(2)$ is the 1-simplex with extreme elements $I_2$ and $P(1,2)$, the extreme elements of $\pi(\mathbf{DS}(2))$ are completely characterized by Corollary 4.2. Indeed, $\mathcal{E}(\pi(\mathbf{DS}(2)))$ has precisely four elements, namely, the identity map and the maps $X \mapsto P(1,2)X$, $X \mapsto \mathrm{tr}(J_2X)I_2/2$, and $X \mapsto \mathrm{tr}(J_2X)P(1,2)/2$. So each extreme element of $\pi(\mathbf{DS}(2))$ is of one of the types described in Proposition 4.3.

It is easy to see that the rank-one elements of $\pi(\mathbf{DS}(n))$ are precisely those of the form $X \mapsto (\mathrm{tr}\ J_nX)A/n$, where $A$ is some fixed element of $\mathbf{DS}(n)$, and when $A \in \mathbf{P}(n)$ we obtain an extreme element of $\pi(\mathbf{DS}(n))$, which is the case described in Proposition 4.3(b). Extreme elements of $\pi(\mathbf{DS}(n))$ of the forms (c), (d) given in Proposition 4.3 were suggested by Bob Grone at the 1999 Linear Preserver Workshop held at Lisbon when the first author gave a talk on a preliminary version of this paper. They suggest elements of $\pi(C)$, where $C$ is a compact convex set for which $0 \notin \mathrm{aff}\, C$, that come from nonnegative linear functionals on $C$, which we have described in above.

Note that $\mathbf{DS}(3)$, unlike $\mathbf{DS}(2)$, is not a simplex. The characterization of extreme linear preservers of $\mathbf{DS}(3)$ is more difficult. The extreme elements of $\mathbf{DS}(3)$ satisfy (up to multiples) precisely one linear relation. Let $P_1 = I_3$, $P_2 = P(1,3,2)$, $P_3 = P(1,2,3)$, $P_4 = P(2,3)$, $P_5 = P(1,2)$ and $P_6 = P(1,3)$, so that $P_i$ correspond to an even permutation if $i = 1,2,3$

and corresponds to an odd permutation if $i = 4, 5, 6$. We used the cdd program, which is available at

http://www.ifor.math.ethz.ch/ifor/staff/fukuda/cdd_home/cd.html,

developed by Komei Fukuda to find all the extreme points of a system of linear inequalities $Ax \le b$. We have used it to determine the extreme elements of $\pi(\mathbf{DS}(3))$. It turns out that there are 41460 extreme elements and they can be divided into 3 types:

1. all elements of $T(\mathbf{P}(3))$ are permutation matrices;
2. all elements of $T(\mathbf{P}(3))$ have entries among 0, 1/2, 1;
3. all elements of $T(\mathbf{P}(3))$ have entries among 0, 1/3, 2/3, 1,

We have the following analysis of the extreme elements of $\pi(\mathbf{DS}(3))$.

We say that two linear maps $T$ and $U$, each maps span($\mathbf{DS}(3)$) into itself, are equivalent if

$$U(X) = P_2 f_2(T(P_1 f_1(X) Q_1)) Q_2$$

for all $X$, where $P_1, P_2, Q_1, Q_2 \in \mathbf{P}(3)$ and each of $f_1, f_2$ is either the identity map or the transpose map.

The "meaning" of the above equivalence relation is that one may treat two linear maps which map $\mathbf{DS}(3)$ into itself as equivalent if one of these two maps can be obtained from the other by re-labeling the vertices in the domain $\mathbf{DS}(3)$ and/or in the codomain $\mathbf{DS}(3)$, where the re-labeling is done through the transformation $X \mapsto P_i f_i(X) Q_i$. As this kind of re-labeling results in $6 \times 2 \times 6 = 72$ different maps, an equivalence class of a linear map which maps $\mathbf{DS}(3)$ into itself may have as many as $72 \times 72 = 5184$ elements (linear maps).

We found that the 41460 extreme linear preservers are partitioned into only 29 equivalence classes. The largest number of elements in these equivalence classes is 2592 (and there are 10 such equivalence classes), and the smallest number is 6 (with only 1 such class, the elements of which map all permutation matrices to one particular permutation matrix). As expected, due to symmetry, the number of elements in all these equivalence classes are factors of 5184 (6, 72, 108, 162, 216, 648, 1296, 2592).

In the following table, each row represents one equivalence class. For each row, the first 6 entries represent a representative of the equivalence class in the following way:

Let $P_1, \ldots, P_6$ have the same meanings as before. Suppose $T$ is the map represented by the first 6 entries $a(1), ..., a(6)$ of that row. Here each $a(i)$ is a 4-digit number $p(i)q(i)r(i)s(i)$ (with 0's appended on the left if necessary); for instance, $a(i) = 224 (= 0224)$ means $p(i) = 0, q(i) = 2, r(i) = 2, s(i) = 4$. Then

$$T(P_i) = \begin{pmatrix} p(i)/6 & q(i)/6 & * \\ r(i)/6 & s(i)/6 & * \\ * & * & * \end{pmatrix}$$

for $i = 1, \ldots, 6$, where "$*$" is determined from the constraint that $T(P_i)$ is doubly stochastic. For example, the first 6 entries in the 4th row of the table are 6006 6006 330 6006 6003 333. This gives a representative $T$ of the 4-th equivalence class which is defined by $T(P_1) = I_3 = T(P_2) = T(P_4)$

$$T(P_3) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}, \text{ etc.}$$

The 7-th entry of the row is the ordinal number of the equivalence class, whereas the 8-th entry is the number of elements in that class (e.g., the 4-th equivalence class contains 648 different extreme elements). Here is the table:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6006 | 6006 | 6006 | 6006 | 6006 | 6006 | 1 | 6 |
| 6006 | 6006 | 6000 | 6006 | 6006 | 6000 | 2 | 162 |
| 6006 | 6006 | 600 | 6006 | 6006 | 600 | 3 | 108 |
| 6006 | 6006 | 330 | 6006 | 6003 | 333 | 4 | 648 |
| 6006 | 6006 | 330 | 6003 | 3333 | 3006 | 5 | 216 |
| 6006 | 6003 | 333 | 6006 | 3303 | 3033 | 6 | 1296 |
| 6006 | 6003 | 330 | 6006 | 6000 | 333 | 7 | 1296 |
| 6006 | 6003 | 330 | 6006 | 3330 | 3003 | 8 | 1296 |
| 6006 | 6003 | 330 | 6000 | 3333 | 3006 | 9 | 1296 |
| 6006 | 6000 | 660 | 6006 | 6000 | 660 | 10 | 648 |
| 6006 | 6000 | 442 | 4220 | 4204 | 4024 | 11 | 2592 |
| 6006 | 6000 | 333 | 6006 | 3303 | 3030 | 12 | 2592 |
| 6006 | 6000 | 333 | 6003 | 3303 | 3033 | 13 | 1296 |
| 6006 | 4220 | 4002 | 6000 | 4204 | 4024 | 14 | 648 |
| 6006 | 4220 | 420 | 6000 | 4204 | 442 | 15 | 2592 |
| 6006 | 4220 | 224 | 6006 | 4204 | 240 | 16 | 2592 |
| 6006 | 4220 | 224 | 6006 | 2402 | 2042 | 17 | 1296 |
| 6006 | 3330 | 3003 | 6006 | 3303 | 3030 | 18 | 2592 |
| 6006 | 3330 | 330 | 6003 | 3003 | 660 | 19 | 2592 |
| 6006 | 3330 | 330 | 6000 | 3006 | 660 | 20 | 1296 |
| 6006 | 3330 | 303 | 6006 | 3303 | 330 | 21 | 2592 |
| 6006 | 3330 | 303 | 6006 | 3033 | 600 | 22 | 2592 |
| 6006 | 3303 | 60 | 6003 | 3033 | 333 | 23 | 2592 |
| 6006 | 3300 | 60 | 6003 | 3030 | 333 | 24 | 1296 |
| 6006 | 3300 | 60 | 3330 | 3033 | 3003 | 25 | 1296 |
| 6006 | 2402 | 60 | 4220 | 4024 | 224 | 26 | 2592 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6006 | 600 | 60 | 6006 | 600 | 60 | 27 | 72 |
| 6006 | 600 | 60 | 6003 | 333 | 330 | 28 | 1296 |
| 6006 | 600 | 60 | 6000 | 660 | 6 | 29 | 72 |

We would like to make some observations in connection with the above table.

Among the 29 classes, only the first, second, third, 10-th, 27-th, and 29-th classes, contain linear preservers $T$ of $\mathbf{DS}(3)$ satisfying $T(\mathbf{P}(3)) \subseteq \mathbf{P}(3)$. So, many extreme maps of $\pi(\mathbf{DS}(3))$ are not of the forms given in Proposition 4.3.

The 29-th equivalence class, i.e., the last equivalence class, corresponds to all the "onto" maps of $\pi(\mathbf{DS}(3))$. In fact, they are the only extreme elements of $\pi(\mathbf{DS}(3))$ which can be represented in the form $X \mapsto PXQ$ or $PX^tQ$, where $P, Q \in \mathbf{P}(3)$. One readily checks that in all other cases, $T(\mathbf{DS}(3))$ does not contain all the extreme points of $\mathbf{DS}(3)$.

Since the representative of each equivalence class sends $P_1$ to itself, we find that each extreme linear preserver of $\mathbf{DS}(3)$ maps at least one extreme element to an extreme element. This prompts us to ask:

**Question 4.4** *Is it true that for each positive integer $n$, every extreme element in $\pi(\mathbf{DS}(n))$ sends at least one permutation matrix to a permutation matrix?*

By and large, our present understanding of $\pi(\mathbf{DS}(n))$, or $\pi(\mathbf{DS}(3))$ in particular, is very limited.

# 5 Remarks and related questions

In this section, we mention some remarks and questions related to our study.

First, it is interesting to note that the proof of Theorem 3.2 is shorter than that of Theorem 2.2. One possible approach to prove Theorem 2.2 is to show that every linear map on span($\mathbf{DS}(n)$) mapping $\mathbf{DS}(n)$ (or $\mathbf{P}(n)$) onto itself can be extended to a linear map on $\mathbb{R}^{n\times n}$ mapping $\mathbf{DsS}(n)$ (respectively, $\mathbf{sP}(n)$) onto itself, and then apply Theorem 3.2. Similarly, one can try to prove Theorem 3.4 via Theorem 3.6. However, we have not been able to do these.

Another direction of research related to our problems is to study group and semi-group preservers related to $\mathbf{P}(n)$ and $\mathbf{sP}(n)$ preservers. Note that our result on linear maps mapping $\mathbf{P}(n)$ onto itself resembles many linear preserver results on matrix groups, see [5, Chapter 3]. For example, Wei [10] showed that a linear map $T$ on $\mathbb{R}^{n\times n}$ mapping $\mathbf{O}(n)$ onto itself, where $\mathbf{O}(n)$ denotes the orthogonal group, is of the form $X \mapsto PXQ$ or $X \mapsto PX^tQ$ for some $P, Q \in \mathbf{O}(n)$. However, if one replaces the onto assumption by into, then there are low-rank linear preservers of $\mathbf{O}(n)$ when $n = 2, 4, 8$. By Proposition 4.3, it is easy to construct low-rank linear preservers of $\mathbf{P}(n)$.

One may also consider linear maps on span($\mathbf{P}(n)$) mapping $\mathbf{P}(n)$ to a subgroup of $\mathbf{P}(n)$, or mapping a subgroup of $\mathbf{P}(n)$ onto itself. A natural candidate is the alternate group $\mathbf{A}(n)$. We have obtained some partial results on this problem. We exclude the trivial case when $n = 1, 2$.

(1) The matrices in $\mathbf{A}(3)$ are linearly independent, and hence $\dim(\text{span}(\mathbf{A}(3))) = 3 < 5 = \dim(\text{span}(\mathbf{P}(3)))$. A linear map $T$ on span($\mathbf{A}(3)$) strongly preserves $\mathbf{A}(3)$ if and only if $T$ permutes the three even permutation matrices.

(2) Suppose $n > 3$. Then $\text{span}(\mathbf{A}(n)) = \text{span}(\mathbf{P}(n))$. Suppose a linear map $T$ on span $\mathbf{A}(n)$ is of the form $T(X) = PXQ$ or $T(X) = PX^tQ$, where $P,\ Q \in \mathbf{P}(n)$ such that $PQ \in \mathbf{A}(n)$. Then $T$ is a linear strong preserver of $\mathbf{A}(n)$.

It would be nice if all the linear strong preservers of $\mathbf{A}(n)$ are of the form described in (2) when $n > 3$. However, this is not true in general as can be shown in the following result on $\mathbf{A}(4)$.

(3) Let $n = 4$, and let

$$S_1 = \{I_4, P(1,2)P(3,4),\ P(1,3)P(2,4),\ P(1,4)P(2,3)\},$$

$$S_2 = \{P(1,2,3),\ P(1,3,4),\ P(1,4,2),\ P(2,4,3)\}$$

and $$S_2^t = \{P(1,3,2),\ P(1,4,3),\ P(1,2,4),\ P(2,3,4)\}.$$

If $\psi : \mathbf{A}(4) \to \mathbf{A}(4)$ is a bijective mapping that satisfies:

$$\psi(S_i) = S_{j_i} \qquad \text{with} \qquad \{j_1, j_2, j_3\} = \{1, 2, 3\}, \tag{5.1}$$

then $\psi$ can be extended to a unique linear strong preserver of $\mathbf{A}(4)$. Conversely, if $\psi : \text{span}(\mathbf{A}(4)) \to \text{span}(\mathbf{A}(4))$ is a linear strong preserver of $\mathbf{A}(4)$, then $\psi$ satisfies (5.1).

An even more challenging problem is to study linear (not necessarily strong) preservers of $\mathbf{A}(n)$. Also, not much is known about semi-group preserving maps related to $\mathbf{sP}(n)$.

Note that $\mathbf{O}(n)$ and $\mathbf{P}(n)$ are reflection groups (groups generated by reflections). In [4], the authors studied linear strong preservers of finite reflection groups and also other related problems.

Since the mid-seventies, a fair amount of work has been done on the problem of determining the extreme elements of $\pi(K_1, K_2)$ for proper (i.e. closed, pointed, full, convex) cones $K_1,\ K_2$ in possibly different finite dimensional real linear spaces, where $\pi(K_1, K_2)$ denotes the set of all linear maps $T$ that satisfy $T(K_1) \subseteq K_2$. By now the problem is completely solved only in a few special cases when the underlying cones $K_1$ and $K_2$ are simple enough. The general problem seems to be far from being resolved. Our Proposition 4.1 and Corollary 4.2 have counterparts for the proper cone case, but the results are not exactly the same. For the interested reader, consult the research-expository paper [9].

**Note added in proof.**

Very recently, Hanley Chiang and the first author have characterized linear operators $\phi$ satisfying $\phi(\mathbf{A}(n)) = \mathbf{A}(n)$. In particular, they showed that if $n = 4$, such a map must be of the form described in (3); if $n \geq 5$, such a map must be of the form described in (2).

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