Linear preservers of Tensor product of Unitary Orbits, and Product Numerical Range

Chi-Kwong Li*, Yiu-Tung Poon†, and Nung-Sing Sze‡

Abstract

It is shown that the linear group of automorphism of Hermitian matrices which preserves the tensor product of unitary orbits is generated by *natural* automorphisms: change of an orthonormal basis in each tensor factor, partial transpose in each tensor factor, and interchanging two tensor factors of the same dimension. The result is then applied to show that automorphisms of the product numerical ranges have the same structure.

2000 Mathematics Subject Classification. 15A86, 15A69, 15A60, 15B57, 46N50.
Keywords. Unitary orbits, automorphism, quantum states, density matrices, partial transpose, product numerical range.

Dedicated to Professors Abraham Berman, Moshe Goldberg, and Raphael Loewy

1 Introduction

Let M_n be the set of $n \times n$ complex matrices, and H_n be the set of Hermitian matrices in M_n. In quantum physics, quantum states of a system with n physical states are represented as density matrices A in H_n, i.e., A is positive semi-definite with trace one; see [7]. Let $C \in H_m$ and $D \in H_n$ be density matrices. They may be changed by quantum operations, or they may be put in different bases for easy measurement. In closed systems, these correspond to unitary similarity transforms. Hence, it is interesting to consider the unitary similarity orbits of these matrices, namely,

$$\mathcal{U}(C) = \{UCU^* : U \in M_m \text{ is unitary}\} \quad \text{and} \quad \mathcal{U}(D) = \{VDV^* : V \in M_n \text{ is unitary}\}.$$

If there is no influence from the external environment, the joint system described by the states $X \in \mathcal{U}(C)$ and $Y \in \mathcal{U}(D)$ is represented by $X \otimes Y$. When C and D are pure states, i.e., both C and D are rank one orthogonal projections, then $\mathcal{U}(C) \otimes \mathcal{U}(D)$ contains all states of the form

*Department of Mathematics, College of William & Mary, Williamsburg, Virginia 23187-8795, USA. This research was done while he was visiting the Hong Kong University of Science and Technology in 2011 under the support of a Fulbright Fellowship. Li is an honorary professor of the University of Hong Kong, the Taiyuan University of Technology, and the Shanghai University. (ckli@math.wm.edu)

†Department of Mathematics, Iowa State University, Ames, Iowa 50011, USA. (ytpoon@iastate.edu)

‡Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong. (raymond.sze@inet.polyu.edu.hk)

$X \otimes Y$, where $X \in H_m$ and $Y \in H_n$ are pure states, and the convex hull of $\mathcal{U}(C) \otimes \mathcal{U}(D)$, denoted by $\mathcal{S}(C,D) = \mathrm{conv}\,\{\mathcal{U}(C) \otimes \mathcal{U}(D)\}$, becomes the set of all separable bipartite states; see [3].

In [1], we show that linear automorphisms on H_{mn} leaving invariant the set $\mathcal{U}(C) \otimes \mathcal{U}(D)$ have the same structure as those leaving invariant the set $\mathcal{S}(C,D)$ when C and D are pure states. Such an linear automorphism Ψ has the form

$$(1)\ A \otimes B \mapsto \psi_1(A) \otimes \psi_2(B) \quad \text{or} \quad (2)\ A \otimes B \mapsto \psi_2(B) \otimes \psi_1(A),$$

where for $j = 1, 2$, ψ_j has the form

$$X \mapsto U_j^* X U_j \quad \text{or} \quad X \mapsto U_j^* X^t U_j$$

for some unitary $U_1 \in M_m$ and $U_2 \in M_n$.

The purpose of this paper is to refine the above result, and characterize linear automorphisms Ψ on H_{mn} or M_{mn} such that

$$\Psi(\mathcal{U}(C) \otimes \mathcal{U}(D)) = \mathcal{U}(C) \otimes \mathcal{U}(D) \quad \text{and / or} \quad \Psi(\mathcal{S}(C,D)) = \mathcal{S}(C,D),$$

where $C \in H_m$ and $D \in H_n$ are density matrices.

In connection to $\mathcal{U}(C) \otimes \mathcal{U}(D)$, consider the (C,D)-product numerical range of an $(mn) \times (mn)$ matrix defined by

$$W^{\otimes}_{C,D}(T) = \{\mathrm{tr}\,(TZ) : Z \in \mathcal{U}(C) \otimes \mathcal{U}(D)\},$$

which is a generalization of the classical numerical range (see [2]) and is a useful tool for studying quantum information science introduced in [6]. We will also characterize linear maps Ψ satisfying

$$W^{\otimes}_{C,D}(\Psi(T)) = W^{\otimes}_{C,D}(T) \qquad \text{for all matrices } T \in M_{mn}.$$

Note that when $D = I_n/n$, we can consider the composite map $\mathrm{tr}_2 \circ \Psi$, where tr_2 is the linear map such that $\mathrm{tr}_2(A \otimes B) = (\mathrm{tr}\,B)A$ for $A \otimes B \in M_m \otimes M_n$ known as the partial trace operator with respect to the second system. Then the problems reduce to the study of linear preservers of $\mathcal{U}(C)$ and the linear preservers of the C-numerical range $W_C(T)$; see [4] and its references.

To avoid degenerate cases, we always assume that C and D are non-scalar matrices in our discussion. Furthermore, we use the usual inner product $(X,Y) = \mathrm{tr}\,(XY^*)$ for two complex matrices of the same size. Also, to specify a linear map on H_{mn} or M_{mn}, it suffices to (and we often will) specify only the image of elements of the form $A \otimes B$.

2 Results and proofs

Consider the following sets of linear maps on complex or Hermitian matrices.

$\mathcal{L}(C)$: the set of operators mapping $\mathcal{U}(C)$ onto itself.

$\mathcal{L}(D)$: the set of operators mapping $\mathcal{U}(D)$ onto itself.

$\mathcal{L}(C,D)$: the set of operators mapping $\mathcal{U}(C)$ onto $\mathcal{U}(D)$.

By the result in [5], operators in $\mathcal{L}(C)$ have the form

(1) $A \mapsto UAU^*$ or $A \mapsto UA^tU^*$ for some unitary $U \in M_m$,

(2) $A \mapsto (2\mathrm{tr}\, A/m)I_m - UAU^*$ or $A \mapsto (2\mathrm{tr}\, A/m)I_m - UA^tU^*$ for some unitary $U \in M_m$ in case C and $2I/m - C$ have the same eigenvalues.

Similarly, operators in $\mathcal{L}(D)$ have the forms

(3) $B \mapsto VBV^*$ or $B \mapsto VB^tV^*$ for some unitary $V \in M_n$,

(4) $B \mapsto (2\mathrm{tr}\, B/n)I_n - VBV^*$ or $B \mapsto (2\mathrm{tr}\, B/n)I_n - VB^tV^*$ for some unitary $V \in M_n$ in case D and $2I/n - D$ have the same eigenvalues.

For $\mathcal{L}(C,D)$ to be non-empty, we must have $m = n$. If $\mathcal{U}(C) = \mathcal{U}(D)$, i.e., C and D have the same eigenvalues, then $\mathcal{L}(C,D) = \mathcal{L}(C)$ and $\mathcal{L}(C,D)$ consists of operators of the form (1). Otherwise, $2I_m/m - C$ and D have the same eigenvalues, equivalently, $2I_n/n - D$ and C have the same eigenvalues, and $\mathcal{L}(C,D)$ consists of operators of the form (2) described above.

We have the following.

Theorem 2.1 *Let $\Psi : \mathcal{V} \to \mathcal{V}$ be a linear map with $\mathcal{V} \in \{M_{mn}, H_{mn}\}$, and $C \in H_m$ and $D \in H_n$ be non-scalar density matrices. The following are equivalent.*

(a) $\Psi(\mathcal{U}(C) \otimes \mathcal{U}(D)) = \mathcal{U}(C) \otimes \mathcal{U}(D)$.

(b) $\Psi(\mathcal{S}(C,D)) = \mathcal{S}(C,D)$.

(c) *One of the following holds.*

(c.1) *There are $\psi_1 \in \mathcal{L}(C)$ and $\psi_2 \in \mathcal{L}(D)$ such that*

$$\Psi(A \otimes B) = \psi_1(A) \otimes \psi_2(B) \text{ for all } A \otimes B \in H_m \otimes H_n.$$

(c.2) $(m, \mathcal{U}(C)) = (n, \mathcal{U}(D))$*, there are $\psi_1 \in \mathcal{L}(C)$ and $\psi_2 \in \mathcal{L}(D)$ such that*

$$\Psi(A \otimes B) = \psi_2(B) \otimes \psi_1(A) \text{ for all } A \otimes B \in H_m \otimes H_n.$$

(c.3) $(m, \mathcal{U}(2I_m/m - C)) = (n, \mathcal{U}(D))$*, and there are $\psi_1, \psi_2 \in \mathcal{L}(C,D)$ such that*

$$\Psi(A \otimes B) = \psi_2(B) \otimes \psi_1(A) \text{ for all } A \otimes B \in H_m \otimes H_n.$$

In the rest of this section, we always assume that $C \in H_m$ and $D \in H_n$ such that $C \neq I_m/m$ and $D \neq I_n/n$. To prove Theorem 2.1, we first establish some lemmas.

Lemma 2.2 *Given any four distinct elements $X_1 \otimes Y_1, \ldots, X_4 \otimes Y_4$ in $\mathcal{U}(C) \otimes \mathcal{U}(D)$. Suppose*

$$\sum_{j=1}^{4} \alpha_j \left(X_j \otimes Y_j \right) = 0, \tag{1}$$

for some nonzero $\alpha_1, \ldots, \alpha_4 \in \mathbb{R}$ with $\alpha_1 + \cdots + \alpha_4 = 0$. Then either

$$X_1 = X_2 = X_3 = X_4 \quad \textit{or} \quad Y_1 = Y_2 = Y_3 = Y_4.$$

Proof. Without loss of generality, suppose $X_1 \neq X_2$. Then X_1 and X_2 are linearly independent and there is a linear functional $f : H_m \to \mathbb{R}$ such that $f(X_1) = 1$ and $f(X_2) = 0$. Applying the linear map $A \otimes B \mapsto f(A)B$ to equation (1),

$$\alpha_1 Y_1 + \alpha_3 f(X_3) Y_3 + \alpha_4 f(X_4) Y_4 = 0 \quad \Longrightarrow \quad Y_1 = \left(-\alpha_3 f(X_3)/\alpha_1\right) Y_3 + \left(-\alpha_4 f(X_4)/\alpha_1\right) Y_4.$$

Notice that at least one of $f(X_3)$ and $f(X_4)$ is nonzero. Suppose $f(X_3) \neq 0$. Then we must have $Y_1 = Y_3$ as Y_1, Y_3, Y_4 are in $\mathcal{U}(D)$. In this case, we must have $X_1 \neq X_3$. Then there is another linear functional $g : H_m \to \mathbb{R}$ such that $g(X_1) = 1$ and $g(X_2) = g(X_3) = 0$. Applying g to (1),

$$\alpha_1 Y_1 + \alpha_4 g(X_4) Y_4 = 0.$$

Then we have $Y_1 = Y_4$. Taking the partial trace $A \otimes B \mapsto (\operatorname{tr} A)B$ in (1), one gets

$$\alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3 + \alpha_4 Y_4 = 0.$$

Since, $Y_1 = Y_3 = Y_4$, we must have $Y_1 = Y_2$. The result follows. $\square$

For any $A \in M_m$, let $A(i,j)$ be the submatrix of A with row and column indices i and j.

Lemma 2.3 *Suppose $D = \operatorname{diag}(d_1, \ldots, d_n)$ is not a scalar matrix. For any (i,j) pair, let $\mathcal{T}_{ij}(D)$ be the set of matrices in $\mathcal{U}(D)$ obtained from D by replacing $D(i,j)$ by a matrix in H_2 with eigenvalues d_i and d_j.*

(1) *If $d_i \neq d_j$, then for any two distinct matrices $T_1, T_2 \in \mathcal{T}_{ij}(D)$, there are $T_3, T_4 \in \mathcal{T}_{ij}(D)$ such that T_1, T_2, T_3 and T_4 are all distinct and either $T_1 + T_3 = T_2 + T_4$ or $T_1 + T_2 = T_3 + T_4$.*

(2) *For any permutation σ on the index set $\{1, \ldots, n\}$, define $D_\sigma = \operatorname{diag}\left(d_{\sigma(1)}, \ldots, d_{\sigma(n)}\right)$. Then the real linear span of the set $\bigcup\{\mathcal{T}_{ij}(D_\sigma) : \text{permutation } \sigma \text{ and } 1 \leq i < j \leq n\}$ equals H_n.*

Proof. For the first statement, we assume that $(i,j) = (1,2)$, $T_1 = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \oplus \hat{D}$ and $T_2 = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \oplus \hat{D}$ with $\hat{D} = \operatorname{diag}(d_3, \ldots, d_n)$. Consider the following two cases:

Case 1. Suppose $T_2 \neq \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix} \oplus \hat{D}$. Let $T_3 = \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix} \oplus \hat{D}$ and $T_4 = \begin{pmatrix} y_{22} & -y_{12} \\ -y_{21} & y_{11} \end{pmatrix} \oplus \hat{D}$. Then T_1, T_2, T_3 and T_4 are all distinct and $T_1 + T_3 = (d_1 + d_2) I_2 \oplus 2\hat{D} = T_2 + T_4$.

Case 2. Suppose $T_2 = \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix} \oplus \hat{D}$. One can always choose T_3 and $T_4 \in \mathcal{T}_{12}(D)$ so that T_1, T_2, T_3 and T_4 are all distinct and $T_1 + T_2 = (d_1 + d_2)I_2 \oplus 2\hat{D} = T_3 + T_4$.

For the second statement, clearly, the set $\{D_\sigma : \text{permutation } \sigma\}$ spans the set of all diagonal matrices in H_n. Next, for each (r, s) pair, one can find a permutation σ so that $d_{\sigma(r)} \neq d_{\sigma(s)}$ and hence $\mathcal{T}_{rs}(D_\sigma)$ contains two linearly independent matrices with nonzero (r, s) and (s, r) entries. Therefore, the set $\bigcup\{\mathcal{T}_{ij}(D_\sigma) : \text{permutation } \sigma \text{ and } 1 \leq i < j \leq n\}$ clearly spans H_n. $\square$

Lemma 2.4 *Suppose $C \in H_m$ and $D \in H_n$ are non-scalar density matrices. Let $\Psi : \mathcal{V} \to \mathcal{V}$ be a linear map with $\mathcal{V} \in \{M_{mn}, H_{mn}\}$ such that $\Psi(\mathcal{U}(C) \otimes \mathcal{U}(D)) = \mathcal{U}(C) \otimes \mathcal{U}(D)$. Then one of the following holds.*

(1) *For every $X \in \mathcal{U}(C)$ there is $\tilde{X} \in \mathcal{U}(C)$ such that $\Psi(X \otimes \mathcal{U}(D)) = \tilde{X} \otimes \mathcal{U}(D)$; for every $Y \in \mathcal{U}(D)$ there is $\tilde{Y} \in \mathcal{U}(D)$ such that $\Psi(\mathcal{U}(C) \otimes Y) = \mathcal{U}(C) \otimes \tilde{Y}$.*

(2) *$m = n$, for every $X \in \mathcal{U}(C)$ there is $\tilde{X} \in \mathcal{U}(D)$ such that $\Psi(X \otimes \mathcal{U}(D)) = \mathcal{U}(C) \otimes \tilde{X}$; for every $Y \in \mathcal{U}(D)$ there is $\tilde{Y} \in \mathcal{U}(C)$ such that $\Psi(\mathcal{U}(C) \otimes Y) = \tilde{Y} \otimes \mathcal{U}(D)$.*

Proof. Without loss of generality, assume that $m \leq n$ and $D = \operatorname{diag}(d_1, \ldots, d_n)$. For any permutation σ on the index set $\{1, \ldots, n\}$, define $D_\sigma = \operatorname{diag}\left(d_{\sigma(1)}, \ldots, d_{\sigma(n)}\right)$. Fixed a $C_0 \in \mathcal{U}(C)$. We first claim that each D_σ with $d_{\sigma(i)} \neq d_{\sigma(j)}$, either

(i) there is a $\tilde{C}_0 \in \mathcal{U}(C)$ such that $\Psi\left(C_0 \otimes \mathcal{T}_{ij}(D_\sigma)\right) \subseteq \tilde{C}_0 \otimes \mathcal{U}(D)$; or

(ii) there is a $\tilde{C}_0 \in \mathcal{U}(D)$ such that $\Psi\left(C_0 \otimes \mathcal{T}_{ij}(D_\sigma)\right) \subseteq \mathcal{U}(C) \otimes \tilde{C}_0$.

Suppose T_1 and $T_2 \in \mathcal{T}_{ij}(D_\sigma)$ are distinct. By Lemma 2.3(1), there exist T_3 and $T_4 \in \mathcal{T}_{ij}(D_\sigma)$ such that T_1, T_2, T_3 and T_4 are distinct and either $T_1 + T_3 = T_2 + T_4$ or $T_1 + T_2 = T_3 + T_4$. Let $\Psi(C_0 \otimes T_i) = X_i \otimes Y_i$ for some $X_i \in \mathcal{U}(C)$ and $Y_i \in \mathcal{U}(D)$, $i = 1, \ldots, 4$. Then $X_1 \otimes Y_1 + X_3 \otimes Y_3 = X_2 \otimes Y_2 + X_4 \otimes Y_4$ or $X_1 \otimes Y_1 + X_2 \otimes Y_2 = X_3 \otimes Y_3 + X_4 \otimes Y_4$. By Lemma 2.2, we have either $X_1 = \cdots = X_4$ or $Y_1 = \cdots = Y_4$. As T_1 and T_2 are arbitrary matrices in $\mathcal{T}_{ij}(D_\sigma)$, the claim holds.

Suppose first $\Psi(C_0 \otimes \mathcal{T}_{12}(D)) \subseteq \tilde{C}_0 \otimes \mathcal{U}(D)$ for some $\tilde{C}_0 \in \mathcal{U}(C)$. In this case, we will show that

$$\{\Psi(C_0 \otimes D_\sigma) : \text{permutation } \sigma\} \subseteq \tilde{C}_0 \otimes \mathcal{U}(D). \tag{2}$$

Once this is proven, with Lemma 2.3 and the claims (i)-(ii), one can conclude that $\Psi(C_0 \otimes \mathcal{U}(D)) \subseteq \tilde{C}_0 \otimes \mathcal{U}(D)$. Applying the argument to Ψ^{-1} on the set $\tilde{C}_0 \otimes \mathcal{U}(D)$, we see that $\Psi^{-1}(\tilde{C}_0 \otimes \mathcal{U}(D)) \subseteq C_0 \otimes \mathcal{U}(D)$. Thus, $\Psi(C_0 \otimes \mathcal{U}(D)) = \tilde{C}_0 \otimes \mathcal{U}(D)$.

To prove the inclusion (2), let $D' = \operatorname{diag}(d_2, d_1, d_3, \ldots, d_n)$. Notice that $\{D, D'\} \subseteq \mathcal{T}_{12}(D)$. By claim (i),

$$\Psi(C_0 \otimes D) = \tilde{C}_0 \otimes Y \quad \text{and} \quad \Psi(C_0 \otimes D') = \tilde{C}_0 \otimes Y' \quad \text{for some distinct } Y, Y' \in \mathcal{U}(D). \tag{3}$$

We consider the following two cases.

Case 1 Suppose $d_{\sigma(1)} \neq d_{\sigma(2)}$. Let $D'_\sigma = \text{diag}\,\big(d_{\sigma(2)}, d_{\sigma(1)}, d_{\sigma(3)}, \ldots, d_{\sigma(n)}\big)$. Then

$$\begin{aligned} & (d_{\sigma(1)} - d_{\sigma(2)})\,\big(\Psi(C_0 \otimes D) - \Psi(C_0 \otimes D')\big) - (d_1 - d_2)\,\big(\Psi(C_0 \otimes D_\sigma) - \Psi(C_0 \otimes D'_\sigma)\big) \\ = \; & \Psi\,\big((d_{\sigma(1)} - d_{\sigma(2)})(C_0 \otimes (D - D')) - (d_1 - d_2)(C_0 \otimes (D_\sigma - D'_\sigma))\big) = 0. \end{aligned}$$

Then Lemma 2.2 and (3) imply $\Psi(C_0 \otimes D_\sigma) = \tilde{C}_0 \otimes Y_\sigma$ for some $Y_\sigma \in \mathcal{U}(D)$.

Case 2 Suppose $d_{\sigma(1)} = d_{\sigma(2)}$. Clearly, there is $j \geq 2$ such that $d_{\sigma(j)} \neq d_{\sigma(1)}$. Without loss of generality, we may assume $j = 3$, i.e., $d_{\sigma(3)} \notin \{d_{\sigma(1)}, d_{\sigma(2)}\}$. Let $D'' = \text{diag}\,(d_1, d_3, d_2, \ldots, d_n)$ and $D''_\sigma = \text{diag}\,\big(d_{\sigma(1)}, d_{\sigma(3)}, d_{\sigma(2)}, \ldots, d_{\sigma(n)}\big)$. By Case 1, $\Psi(C_0 \otimes D''_\sigma) = \tilde{C}_0 \otimes Y''_\sigma$ for some $Y''_\sigma \in \mathcal{U}(D)$. Observe that

$$(d_{\sigma(2)} - d_{\sigma(3)})\,\big(\Psi(C_0 \otimes D) - \Psi(C_0 \otimes D'')\big) - (d_2 - d_3)\,\big(\Psi(C_0 \otimes D_\sigma) - \Psi(C_0 \otimes D''_\sigma)\big) = 0.$$

With Lemma 2.2 and (3), one can conclude that $\Psi(C_0 \otimes D_\sigma) = \tilde{C}_0 \otimes Y_\sigma$ for some $Y_\sigma \in \mathcal{U}(D)$. Therefore, the inclusion (2) holds.

Next suppose $\Psi(C_0 \otimes \mathcal{T}_{12}(D)) \subseteq \mathcal{U}(C) \otimes \tilde{C}_0$ for some $\tilde{C}_0 \in \mathcal{U}(D)$. By a similar argument, one can show that $\Psi(C_0 \otimes \mathcal{U}(D)) \subseteq \mathcal{U}(C) \otimes \tilde{C}_0$. Then Ψ induces an injective map from span $\{C_0 \otimes \mathcal{U}(D)\} = C_0 \otimes M_n$ to span $\{\mathcal{U}(C) \otimes \tilde{C}_0\} = M_m \otimes \tilde{C}_0$. Since we assume that $m \leq n$, we conclude that $m = n$. Applying the argument to Ψ^{-1} on the set $\mathcal{U}(C) \otimes \tilde{C}_0$, we see that $\Psi(C_0 \otimes \mathcal{U}(D)) = \mathcal{U}(C) \otimes \tilde{C}_0$.

From the above argument, one see that for each $C_0 \in \mathcal{U}(C)$, either $\Psi(C_0 \otimes \mathcal{U}(D)) = \tilde{C}_0 \otimes \mathcal{U}(D)$ or $\Psi(C_0 \otimes \mathcal{U}(D)) = \mathcal{U}(C) \otimes \tilde{C}_0$. Now, we claim that one of the following holds.

(I) For every $X \in \mathcal{U}(C)$ there is $\tilde{X} \in \mathcal{U}(C)$ such that $\Psi(X \otimes \mathcal{U}(D)) = \tilde{X} \otimes \mathcal{U}(D)$.

(II) For every $X \in \mathcal{U}(C)$, there is $\tilde{X} \in \mathcal{U}(D)$ such that $\Psi(X \otimes \mathcal{U}(D)) = \mathcal{U}(C) \otimes \tilde{X}$.

To see this, consider any distinct $X_1, X_2 \in \mathcal{U}(C)$. Suppose $\Psi(X_1 \otimes \mathcal{U}(D)) = \tilde{X}_1 \otimes \mathcal{U}(D)$ and $\Psi(X_2 \otimes \mathcal{U}(D)) = \mathcal{U}(C) \otimes \tilde{X}_2$ for some $\tilde{X}_1 \in \mathcal{U}(C)$ and $\tilde{X}_2 \in \mathcal{U}(D)$. Then

$$\Psi(X_1 \otimes \mathcal{U}(D)) \cap \Psi(X_2 \otimes \mathcal{U}(D)) = \{\tilde{X}_1 \otimes \tilde{X}_2\}.$$

But this contradicts the fact that Ψ is bijective and the two sets $X_1 \otimes \mathcal{U}(D)$ and $X_2 \otimes \mathcal{U}(D)$ are disjoint.

Now, suppose $D_0 \in \mathcal{U}(D)$. We can apply similar arguments to conclude that either

(i') $\Psi(\mathcal{U}(C) \otimes D_0) = \mathcal{U}(C) \otimes \tilde{D}_0$ for some $\tilde{D}_0 \in \mathcal{U}(D)$, or

(ii') $\Psi(\mathcal{U}(C) \otimes D_0) \subseteq \tilde{D}_0 \otimes \mathcal{U}(D)$ for some $\tilde{D}_0 \in \mathcal{U}(C)$.

Note that in (ii'), we cannot get $m = n$ and the set equality as before because we assume that $m \le n$.

We will show that if (I) holds then (i') holds. Assume the contrary that (I) and (ii') hold. We can find $X_1 \in \mathcal{U}(C)$ such that $\Psi(X_1 \otimes \mathcal{U}(D)) = \tilde{X}_1 \otimes \mathcal{U}(D)$ with $\tilde{X}_1 \neq \tilde{D}_0$. Then

$$\Psi(X_1 \otimes D_0) \in \Psi(X_1 \otimes \mathcal{U}(D)) \cap \Psi(\mathcal{U}(C) \otimes D_0) \subseteq \left(\tilde{X}_1 \otimes \mathcal{U}(D)\right) \cap \left(\tilde{D}_0 \otimes \mathcal{U}(D)\right) = \emptyset,$$

a contradiction. Thus, if (I) holds, then (i') holds.

Similarly, if (II) holds we can show that (i') cannot hold. Thus, we must have condition (ii') with the additional conclusion that the set equality $\Psi(\mathcal{U}(C) \otimes D_0) = \tilde{D}_0 \otimes \mathcal{U}(D)$. Now for any $Y \in \mathcal{U}(D)$, we can show that $\Psi(\mathcal{U}(C) \otimes Y) = \mathcal{U}(C) \otimes \tilde{Y}$ for some $\tilde{Y} \in \mathcal{U}(D)$, or $\Psi(\mathcal{U}(C) \otimes Y) = \tilde{Y} \otimes \mathcal{U}(D)$ for some $\tilde{Y} \in \mathcal{U}(C)$, depending on (i') or (ii') holds. The desired result follows. $\square$

Proof of Theorem 2.1 Since $\mathcal{U}(C) \otimes \mathcal{U}(D)$ is the set of extreme points of $\mathcal{S}(C, D)$, we have (a) $\Leftrightarrow$ (b). Clearly, (c) $\Rightarrow$ (a).

Suppose (a) holds. By Lemma 2.4, either (1) or (2) holds. Suppose (1) holds. Let $\psi_1 = \mathrm{tr}_2 \circ \Psi$ and $\psi_2 = \mathrm{tr}_1 \circ \Psi$, where tr_1 and tr_2 are the partial traces given by $\mathrm{tr}_1(A \otimes B) = \mathrm{tr}(A)B$ and $\mathrm{tr}_2(A \otimes B) = \mathrm{tr}(B)A$. It follows that (c.1) holds. Similarly, we have either (c.2) or (c.3) if (2) holds.. $\square$

By Theorem 2.1, we can deduce the following.

Theorem 2.5 *Let $\Psi : \mathcal{V} \to \mathcal{V}$ be a linear map with $\mathcal{V} \in \{M_{mn}, H_{mn}\}$, and $C \in H_m$ and $D \in H_n$ be non-scalar density matrices. The following are equivalent.*

(a) $W^{\otimes}_{C,D}(\Psi(T)) = W^{\otimes}_{C,D}(T)$ *for all* $T \in \mathcal{V}$.

(b) $\mathrm{conv}\left(W^{\otimes}_{C,D}(\Psi(T))\right) = \mathrm{conv}\left(W^{\otimes}_{C,D}(T)\right)$ *for all* $T \in \mathcal{V}$.

(c) Ψ *has the form described in Theorem 2.1(c).*

Proof. The implications (c) $\Rightarrow$ (a) $\Rightarrow$ (b) are clear. Suppose (b) holds. Note that

$$\mathrm{conv}\,\{W^{\otimes}_{C,D}(T)\} = \{\mathrm{tr}\,(TZ) : Z \in \mathcal{S}(C, D)\}.$$

Thus the dual map Ψ^* satisfies $\Psi^*(\mathcal{S}(C, D)) = \mathcal{S}(C, D)$ and has the form described in Theorem 2.1 (c). One readily checks that the dual map of such a map has the same form. The result follows. $\square$

Remark 2.6 One may further extend the results to multi-partite systems $\mathcal{U}(C_1) \otimes \cdots \otimes \mathcal{U}(C_k)$ using techniques similar to those in [1] and the following extension of Lemma 2.2.

If four distinct elements $X_1, X_2, X_3, X_4 \in \mathcal{U}(C_1) \otimes \cdots \otimes \mathcal{U}(C_k)$ satisfy $\alpha_1 X_1 + \cdots + \alpha_4 X_4 = 0$ for some nonzero $\alpha_1, \ldots, \alpha_4 \in \mathbb{R}$ summing up to 0, then $X_1, \ldots, X_4$ differ in only one of the tensor factors.

We omit the discussion.

Acknowledgment

Research of Li was supported by USA NSF, HK RGC and the Shanxi 100 talent program. Reserach of Poon was supported by USA NSF. Research of Sze was supported by HK GRC.

References

[1] S. Friedland, C.K. Li, Y.T. Poon and N.S. Sze, The Automorphism Group of Separable States in Quantum Information Theory, Journal of Mathematical Physics, to appear.

[2] P.R. Halmos, A Hilbert Space Problem Book (Graduate Texts in Mathematics), 2nd edition, Springer, New York, 1982.

[3] M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed states: Necessary and sufficient conditions, Physics Letters A 223 (1996), 1-8.

[4] C.K. Li and N.K. Tsing, Duality between some linear preservers problems: The invariance of the C-numerical range, the C-numerical radius and certain matrix sets, Linear and Multilinear Algebra 23 (1988), 353-362.

[5] C.K. Li and N.K. Tsing, Linear maps relating different unitary similarity orbits or different generalized numerical ranges, Linear Algebra Appl. 223/224 (1995), 463-481.

[6] Z. Puchala, P. Gawron, J.A. Miszczak, Ł. Skowronek, M.D. Choi, K. Życzkowski, Product numerical range in a space with tensor product structure, Linear Algebra Appl., to appear. arXiv:1008.3482v1

[7] K. Życzkowski and I. Bengtsson, Geometry of Quantum States, Cambridge University Press, 2006.