Numerical radius, entropy and preservers on quantum states

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Joint with Pro. Jinchuan Hou, Chi-Kwong Li and Zejun Huang

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The talk is based on the following papers.


[3] Kan He, Jinchuan Hou, Entropy-preserving maps on quantum states, preprint, many thanks for many suggestions from Professor Chi-Kwong Li.


Outline
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(I) Numerical radius and quantum states

(II) Convex combinations preserving transformations on quantum states

(III) Transformations on quantum states leaving numerical radius (or quantum entropy) invariant

(IV) Preservers of tensor product operators

(V) Further discussion
(I) Numerical radius and quantum states
(I) Numerical radius and quantum states

Recall that the numerical range of an operator $A$ is the set

$$W(A) = \{ \langle Ax, x \rangle : x \in H, \|x\| = 1 \},$$

and the numerical radius of $A$ is

$$w(A) = \sup\{ |\lambda| : \lambda \in W(A) \}.$$
The $C$-numerical range of $A$ is the set

$$W_C(A) = \{ \text{tr}(CUAU^*) : U \text{ is unitary} \},$$

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Here, $C$ can be chosen as a trace-class operator in infinite dimensional case.

If $C$ is a rank one projection, then $W_C(A) = W(A)$. 
The topic of numerical range and numerical radius plays an important role in Mathematics and Physics.

Thomas Schulte-Herbruggen, Gunther Dirr, Uwe Helmke, Steffen J. Glaser, The significance of the $C$-numerical range and the local $C$-numerical range in quantum control and quantum information, Linear and Multilinear Algebra, Volume 56, Issue 1,2(2008), pages 3-26.

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The numerical radius and $C$-numerical radius of the quantum state are applied into the theory of quantum control of quantum systems.
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\[
UAU^* \rightarrow C.
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Generally speaking, there is no unitary \( U \) such that \( UAU^* = C \). So one task in quantum control is to find points closed to \( C \) in the unitary orbit of \( A \) \( \{UAU^* | U \text{ is unitary}\} \).
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Since $\|C - UAU^*\|_2^2 = \|A\|_2^2 + \|C\|_2^2 - 2\text{Retr}(CUAU^*)$,

maximality of the real part of the $C$-numerical range of $A$ means the minimal Euclidean distance.

So $C$-numerical ranges can be a measure to help us find points in unitary orbit of $A$ with minimal distance to $C$. 
Another measure of difference between $UAU^*$ and $C$ can be developed by the $C$-numerical radius of $A$:

$$\cos(A, C)_U = \frac{|\langle C, UAU^* \rangle|}{\|A\|_2 \|C\|_2} = \frac{|\text{tr}(CUAU^*)|}{\|A\|_2 \|C\|_2}.$$
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The $C$-numerical radius of $A$ is the maximal value of $|\text{tr}(CUAU^*)|$, and means the minimal angle of $UAU^*, C$. 
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For example, quantum channels and quantum operations are completely positive linear maps; quantum gates are unitary operators.

Therefore, it is helpful to know the characterizations of maps on quantum states leaving invariant some important subsets or quantum properties. Such questions have attracted the attention of many researchers.
Let $S(H)$ be the convex set of all quantum states on Hilbert space $H$. 
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**Kadison’s Theorem**  
$\phi$ is a convex isomorphism from $S(H)$ to $S(H)$, i.e. $\phi$ is bijective and for any states $\rho_1, \rho_2$ and any non-negative scalar $p \leq 1$, $\phi$ satisfies

$$
\phi(p\rho_1 + (1-p)\rho_2) = p\phi(\rho_1) + (1-p)\phi(\rho_2)
$$

if and only if there is a unitary or anti-unitary operator $U$ on $H$ such that $\phi(\rho) = U\rho U^*$ for all $\rho \in S(H)$.

We generalize the Kadison’s theorem.

**Theorem 1** [Kan He, Jinchuan Hou, Chi-Kwong Li, J Func. Anal., 264, 464-478] A bijective map $\phi : S(H) \rightarrow S(H)$ satisfies $\phi([\rho_1, \rho_2]) \subseteq [\phi(\rho_1), \phi(\rho_2)]$ for any $\rho_1, \rho_2 \in S(H)$ if and only if there is an invertible bounded linear operator $M \in B(H)$ such that $\phi$ has the form

$$\rho \mapsto \frac{M \rho M^*}{\text{tr}(M \rho M^*)} \quad \text{or} \quad \rho \mapsto \frac{M \rho^T M^*}{\text{tr}(M \rho^T M^*)},$$

where $\rho^T$ is the transpose of $\rho$ with respect to an orthonormal basis.
Observe that $\phi([\rho_1, \rho_2]) \subseteq [\phi(\rho_1), \phi(\rho_2)] \iff$ for any $t \in [0, 1]$, there exists $s \in [0, 1]$ such that $\phi(t\rho_1 + (1 - t)\rho_2) = s\phi(\rho_1) + (1 - s)\phi(\rho_2)$, such a map $\phi$ is called convex combinations preserving map.
Observe that $\phi([\rho_1, \rho_2]) \subseteq [\phi(\rho_1), \phi(\rho_2)] \quad \Leftrightarrow \quad \text{for any } t \in [0, 1], \text{ there exists } s \in [0, 1] \text{ such that } \phi(t\rho_1 + (1 - t)\rho_2) = s\phi(\rho_1) + (1 - s)\phi(\rho_2), \text{ such a map } \phi \text{ is called convex combinations preserving map.}

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Recall that in quantum mechanics, a quantum measurement is described by a collection $\{M_m\}$ of measurement operators acting on the state space $H$ satisfying $\sum_m M_m^* M_m = I$. 


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Recall that in quantum mechanics, a quantum measurement is described by a collection \( \left\{ M_m \right\} \) of measurement operators acting on the state space \( H \) satisfying \( \sum_m M_m^* M_m = I \).

Let \( M_j \) be a quantum measurement operator. If the state of the quantum system is \( \rho \in S(H) \) before the measurement, then the state after the measurement is \( \frac{M_j \rho M_j^*}{\text{tr}(M_j \rho M_j^*)} \) whenever \( M_j \rho M_j^* \neq 0 \).
If $M_j$ is fixed, we get a quantum measurement map $\phi_j$ defined by $\phi_j(\rho) = \frac{M_j \rho M_j^*}{\text{tr}(M_j \rho M_j^*)}$ from the convex subset $S_M(H) = \{\rho : M_j \rho M_j^* \neq 0\}$ of the (convex) set $S(H)$ of states into $S(H)$. If $M_j$ is invertible, then $\phi_j : S(H) \rightarrow S(H)$ is bijective and will be called an invertible measurement map.
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**Physical meaning of Theorem 1:** It follows that an invertible quantum measurement map $\phi$ is represented as a convex combinations preserving bijective map or the composition of such a preserver and the transpose.
Next we turn to convex combinations preserving transformations on unit balls of Hilbert spaces.
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A quantum system can be simulated well by wave functions, which are described by many different ways. Being regarded as square integrable functions with respect to an appropriate measure, wave functions are unit vectors in the Hilbert space $H = L^2(\mu)$. Denote by $B_1(H)$ the closed unit ball of $H$, $W(H)$ the sphere of $B_1(H)$. 
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A quantum system can be simulated well by wave functions, which are described by many different ways. Being regarded as square integrable functions with respect to an appropriate measure, wave functions are unit vectors in the Hilbert space $H = \mathcal{L}^2(\mu)$. Denote by $B_1(H)$ the closed unit ball of $H$, $\mathcal{W}(H)$ the sphere of $B_1(H)$.

For a pure state $x$, a measurement map $\phi$ satisfies $\phi(x) = \frac{Mx}{\|Mx\|}$, where $M$ is a bounded linear operator on $H$ and $\|Mx\| \neq 0$, and if $M$ is invertible, we also call $\phi$ the invertible measurement map of pure states.
Theorem 2 [Kan He, Jinchuan Hou, preprint] Assume \( \phi : \mathcal{B}_1(H) \to \mathcal{B}_1(H) \) is a bijective map, then the following statements are equivalent:

(i) for arbitrary \( t \in [0, 1] \) and \( x, y \in \mathcal{B}_1(H) \), there exists \( s \in [0, 1] \) such that \( \phi(tx + (1-t)y) = s\phi(x) + (1 - s)\phi(y) \);

(ii) there exists an invertible bounded real linear operator \( M \) on \( H \), a nonzero constant real scalar \( r \) and a bounded real linear functional \( h : H \to \mathbb{R} \) such that \( \phi \) has the form

\[
    x \mapsto \frac{Mx}{h(x) + r}, \quad \text{for all } x \in \mathcal{B}_1(H),
\]

and \( h(x) + r = \|Mx\| \) for all unit vectors \( x \).
Quantum gates are unitary operators. Regarding a quantum gate as a map $\phi$ on state spaces, we have

$$\phi(x) = U x,$$

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The following corollary uncovers partially connection between quantum gates and convex isomorphisms on $B_1(H)$.
Theorem 3 [Kan He, Jinchuan Hou, preprint] Assume \( \phi : B_1(H) \to B_1(H) \) is a bijective map, then the following statements are equivalent:

(i) \( \phi \) is a convex isomorphism, i.e., \( \phi(tx + (1 - t)y) = t\phi(x) + (1 - t)\phi(y) \) for arbitrary \( t \in [0,1] \) and \( x, y \in B_1(H) \).

(ii) there exists an invertible isometric real linear operator \( U \) on \( H \) such that \( \phi \) has the form

\[ x \mapsto Ux, \text{ for all } x \in B_1(H). \]
(III) Transformations on quantum states leaving numerical radius (or quantum entropy) invariant
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There have been a great deal of different and active researches on numerical ranges and numerical radii.
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There have been a great deal of different and active researches on numerical ranges and numerical radii.

One of such subjects is to characterize linear or general maps on operator algebras or operator spaces which preserve the numerical range or numerical radius.


J.-L. Cui, J.-C. Hou, Non-linear numerical radius isometries on atomic nest algebras and diagonal algebras, J. Funct. Anal., 206 (2004), 414-


Jinchuan Hou, Kan He, Xiuling Zhang, Nonlinear maps preserving numerical radius of indefinite skew products of operators, Linear Algebra and its Applications, 430 (2009), 2240-2253.
A map $\Phi : S(H) \to S(H)$ preserves numerical radius of convex combinations if $\Phi$ satisfies that

$$w(\lambda \rho + (1 - \lambda) \sigma) = w(\lambda \Phi(\rho) + (1 - \lambda) \Phi(\sigma))$$

for $\rho, \sigma \in S(H)$ and all $\lambda \in [0, 1]$. 
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Here, we devote to characterizing surjective maps on quantum states preserving numerical radius of convex combinations. Such a problem is from discussion with Professor Chi-Kwong Li.
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However, we obtain the following more refined result.
Theorem 4 [Kan He, Jinchuan Hou, preprint] If $\Phi : S(H) \to S(H)$ is a surjective map, then the following statements are equivalent:

(i) $\Phi$ preserves numerical radius of convex combinations of quantum states;

(ii) there exists some $\lambda_0 \in (0, 1)$ such that $w(\lambda_0 \rho + (1 - \lambda_0)\sigma) = w(\lambda_0 \Phi(\rho) + (1 - \lambda_0)\Phi(\sigma))$ for $\rho, \sigma \in S(H)$;

(iii) there exists a unitary or anti-unitary operator $U$ on $H$ such that $\Phi(\rho) = U\rho U^*$ for all $\rho \in S(H)$. 
Remark that since the numerical radius of a quantum state equals to its operator norm and spectral radius, one can obtain the same characterization of maps on quantum states preserving the operator norm (spectral radius) of convex combinations.
We also are interested in characterizing maps on quantum states (positive operators) preserving other kinds of functions.


Yuan Li, Paul Busch, Von Neumann entropy and majorization, arXiv:1304.7442v2

For a quantum state $\rho$, the quantum entropy of $\rho$ is defined as follows:

$$S(\rho) = -\text{tr}(\rho \log_2 \rho).$$
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Also quantum entropy is called von Neumann entropy, and plays an important role in the theory of quantum information.


In finite dimensional case, assume that $\{p_i\}_{i=1}^n$ are eigenvalues of $\rho \in S(H)$ with $\dim H = n < \infty$ and the Shannon entropy

$$H(\{p_i\}_{i=1}^n) = -\sum_{i=1}^n p_i \log_2 p_i$$

then quantum entropy of $\rho$ equals to the Shannon entropy $H(\{p_i\}_{i=1}^n)$, where $0 \log_2 0 = 0$ and $1 \log_2 1 = 0$. 
From the definition of quantum entropy, an elementary discussion show the following interesting conclusions.
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Quantum entropy is non-negative, i.e., $S(\rho) \geq 0$ for arbitrary $\rho \in S(H)$. $S(\rho) = 0$ if and only if $\rho$ is a rank one projection. $S(\rho) = \log_2 n$ if and only if $\rho = \frac{I}{n}$, where $I$ is the identity on $H$. 
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Let $\rho_i$s be quantum states, $i = 1, 2, ..., k$ and $p_i$s are non-negative scalars with $\sum_{i=1}^{k} p_i = 1$, then

$$\sum_{i=1}^{k} p_i S(\rho_i) \leq S(\sum_{i=1}^{k} p_i \rho_i) \leq H(p_i) + \sum_{i=1}^{k} p_i S(\rho_i).$$
Here, we also characterizing surjective maps on quantum states preserving quantum entropy of convex combinations.
Theorem 5 [Kan He, Jinchuan Hou, preprint] Let $S(H)$ be the set of all quantum states on a complex Hilbert space $H$ with $\dim H = n$ and $1 < n < \infty$. For a surjective map $\phi : S(H) \to S(H)$, the following statements are equivalent:

(I) $\phi$ satisfies for arbitrary $\rho, \sigma \in S(H)$ and $t \in [0, 1]$, $S(t\rho + (1 - t)\sigma) = S(t\phi(\rho) + (1 - t)\phi(\sigma))$;

(II) there exists a unitary or anti-unitary operator $U$ such that $\phi(\rho) = U\rho U^*$ for all $\rho \in S(H)$. 
(IV) Preservers of tensor product operators
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A multipartite quantum system is described by the tensor product of Hilbert spaces. For the finite dimensional bipartite system $H_A \otimes H_B$, a state $\rho^{AB} \in \mathcal{S}(H_A \otimes H_B)$ is separable if $\rho^{AB} = \sum_i p_i \rho_i^A \otimes \sigma_i^B$, where $\rho_i^A$ and $\sigma_i^B$ are states on $H_A$, $H_B$ respectively. Otherwise, $\rho^{AB}$ is called an entangled state. Entanglement is one of the most important aspects of quantum information.
Many authors pay their attentions to determining the structure of linear preservers of tensor product operators.


In the finite dimensional bipartite system $H_A \otimes H_B$ with $\dim H_A = m$ and $\dim H_B = n$, let $H_{mn}$ be the space of all $mn \times mn$ Hermitian matrices, $D_{mn}$ the set of all $mn \times mn$ quantum states and $S_{mn}$ the set of all $mn \times mn$ separable states. Denote by $S(\rho)$ the quantum entropy of $\rho \in D_{mn}$ and $\sigma(A)$ the spectrum of $A$. 
In the finite dimensional bipartite system $H_A \otimes H_B$ with $\dim H_A = m$ and $\dim H_B = n$, let $H_{mn}$ be the space of all $mn \times mn$ Hermitian matrices, $D_{mn}$ the set of all $mn \times mn$ quantum states and $S_{mn}$ the set of all $mn \times mn$ separable states. Denote by $S(\rho)$ the quantum entropy of $\rho \in D_{mn}$ and $\sigma(A)$ the spectrum of $A$.

Ajda Fošner, Zejun Huang, Chi-Kwong Li, Nung-Sing Sze show that for a linear map $\phi : H_{mn} \rightarrow H_{mn}$, $\sigma(\phi(A \otimes B)) = \sigma(A \otimes B)$ for $A \in H_m$ and $B \in H_n$ if and only if there exists a unitary matrix $U$ such that $\phi(A \otimes B) = U\phi_1(A) \otimes \phi_2(B)U^*$, where $\phi_j$ is the identity or transposition, $j = 1, 2$.

Ajda Fošner, Zejun Huang, Chi-Kwong Li, Nung-Sing Sze, Linear preservers and quantum information science, Linear and Multilinear algebra, 2013, 61, 1377-1390
The structure of linear maps preserving numerical radius of tensor products of matrices is determined in the following paper.

Theorem 6 [Kan He, Zejun Huang, Jinchuan Hou, Chi-Kwong Li, preprint] Let $\phi : H_{mn} \to H_{mn}$ be a linear map, then the following statements are equivalent:

(I) $\phi(D_{mn}) \subset D_{mn}$, $S(\phi(\rho_{12})) = S(\rho_{12})$ for $\rho_{12} \in S_{mn}$;

(II) $\sigma(\phi(A \otimes B)) = \sigma(A \otimes B)$ for $A \in H_m$ and $B \in H_n$;

(III) there exists a unitary matrix $U$ such that $\phi(A \otimes B) = U\phi_1(A) \otimes \phi_2(B)U^*$, where $\phi_j$ is the identity or transposition, $j = 1, 2$. 

(V) Further discussion
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Determining the structure of $C$-numerical radius preserving maps on quantum states.
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Determining the structure of $C$-numerical radius preserving maps on quantum states.

Determining the structure of more generalized functional values preserving maps on quantum states, for instance, Shur-convex functions (putted forward by and many suggestions from Professor Chi-Kwong Li).
Thanks!