Singular value inequalities for matrices with numerical ranges in a sector

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joint work with Stephen Drury
McGill University
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The quantity

$$\rho_n(A) \equiv \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}|}$$

is called the growth factor of $A$. 
If $A$ is positive definite, then $\rho_n(A) \leq 1$. 

Examples

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If $A$ is strictly diagonally dominant, then $\rho_n(A) \leq 2$. 

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\end{bmatrix},
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then

\[\rho_n(A) = 2^n - 1.\]
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Higham’s conjecture

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Higham’s conjecture. If $A$ is CSPD, then $\rho_n(A) \leq 2$. 
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George, Ikramov and Kucherov [Numer. Linear Algebra Appl. 9 (2002) 107-114] considered a slightly more general class of matrices. They proved that if $A$ is accretive-dissipative (i.e. real and imaginary part in the Cartesian decomposition are positive definite), then $\rho_n(A) \leq 3 \sqrt{2}$.

If $A$ is CSPD, then $\rho_n(A) \leq 3$.

I proved that [Calcolo (2014) to appear] if $A$ is accretive-dissipative, then $\rho_n(A) \leq 4$. If $A$ is CSPD, then $\rho_n(A) \leq 2 \sqrt{2}$. 
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Analysis on $\rho_n(A)$

**Proposition** If $A = (a_{l,j})$ is accretive-dissipative, then

$$\sqrt{2} \max_{l} |a_{ll}| \geq \max_{l \neq j} |a_{lj}|.$$

Suppose $\max_{j,k} |a_{(k)}^{jj}| = |a_{(k_0)^j}|$ for some $j_0, k_0$, then
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Suppose $\max_{j,k} |a_{jj}^{(k)}| = |a_{j_0j_0}^{(k_0)}|$ for some $j_0, k_0$. 
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$$\rho_n(A) \leq \frac{\sqrt{2}|a_{j_0j_0}^{(k_0)}|}{|a_{j_0j_0}|};$$

Fix numbers $k_0$ and $j_0$, where $k_0 \in \{1, \ldots, n-1\}$ and $j_0 \geq k_0 + 1$. Consider the $(k_0+1) \times (k_0+1)$ principal submatrix of $A$, $[A_{k_0u\backslash v^*a}]$.
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We have

$$a_{j_0j_0}^{(k_0)} = a_{j_0j_0} - v^* A_{k_0}^{-1} u.$$
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Equivalently,
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det A \leq 2|a| \det \hat{A}.
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Connection to determinant inequality

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ be an $n \times n$ accretive-dissipative matrix with $A_{22} \ m \times m$. 


$|\det A| \leq 3 |\det A_{11}| \cdot |\det A_{22}|$.

Lin [Linear Algebra Appl. 438 (2013) 2808-2812]:

$|\det A| \leq c |\det A_{11}| \cdot |\det A_{22}|,$
where $c = \begin{cases} \frac{2m}{3}, & \text{if } m \leq \frac{n}{3}; \\ \frac{2n}{2}, & \text{if } \frac{n}{3} < m \leq \frac{n}{2}. \end{cases}$

I conjectured that $|\det A| \leq 2m |\det A_{11}| \cdot |\det A_{22}|$. 
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Define a sector on the complex plane

\[ S_\alpha = \{ z \in \mathbb{C} : \Re z > 0, |\Im z| \leq (\Re z) \tan \alpha \}, \quad \alpha \in [0, \pi/2). \]
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**Basic properties.** Let \( A, B \) be \( n \times n \) complex matrices. If \( W(A), W(B) \subset S_\alpha \), then
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He also conjectured that if $W(A) \subset S_\alpha$, then

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Drury and I [Oper. & Matrices (2014) to appear]: If $W(A) \subset S_\alpha$, then

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A technical lemma: Let $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ be $n \times n$ matrices, where $X_2, Y_2$ are $m \times n$, such that $YX^* = I_n$. Then

$$\lambda_j(X_2X_2^*)\lambda_{m+1-j}(Y_2Y_2^*) \geq 1, \quad j = 1, \ldots, m.$$
**Theorem** (Drury-Lin 2014) For every $n \times n$ complex matrix $A$ such that $W(A) \subset S_\alpha$, it holds

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**Corollary** Let $X, Y$ be $n \times n$ positive semidefinite matrices. Then

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\sigma_j(X + iY) \leq \sqrt{2} \lambda_j(X + Y), \quad j = 1, \ldots, n.
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Proof. $(1 - i)(X + iY) = (X + Y) + i(Y - X)$ has its numerical range in $S_{\frac{\pi}{4}}$. 

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Bhatia-Kittaneh [Linear Algebra Appl. 431 (2009) 1502-1508]: Let $X, Y$ be $n \times n$ positive semidefinite matrices. Then

$$\lambda_j(X + Y) \leq \sqrt{2}\sigma_j(X + iY), \quad j = 1, \ldots, n.$$
Obviously,

\[ a, b \geq 0 \implies |a + bi| \leq a + b. \]
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It is however, not true that for every \( n \times n \) positive semidefinite matrices

\[ \sigma_j(X + iY) \leq \lambda_j(X + Y), \quad j = 1, \ldots, n. \]
Obviously,
\[ a, b \geq 0 \implies |a + bi| \leq a + b. \]

It is however, not true that for every \( n \times n \) positive semidefinite matrices
\[ \sigma_j(X + iY) \leq \lambda_j(X + Y), \quad j = 1, \ldots, n. \]

**Example** Take
\[
X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.
\]

A calculation shows \( \sigma_2(X + iY) \approx 0.4569 > \lambda_2(X + Y) \approx 0.3820. \)
Thank You!