Higher-rank Numerical Range and Quantum Error Correction

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A state of $N$-qubits $Q_1, \ldots, Q_N$ is represented by their tensor products in $M_n$ with $n = 2^N$. 
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A quantum channel $\mathcal{E}$ for states of $N$-qubits corresponds to a **trace preserving completely positive linear maps** $\mathcal{E} : M_n \rightarrow M_n$ of the form

$$\mathcal{E}(\rho) = \sum_{j=1}^{r} E_j \rho E_j^\dagger$$

with

$$\sum_{j} E_j^\dagger E_j = I$$

[Choi (1975)]
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The matrices $E_1, \ldots, E_m$ are interpreted as the error(noise) operators of the quantum channel $\mathcal{E}$. 
Notations

Pauli matrices

\[ \sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]
\[ \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

- \( \sigma_{(j_1, j_2, \ldots, j_N)} := \sigma_{j_1} \otimes \sigma_{j_2} \cdots \otimes \sigma_{j_N}, \quad (j_1, j_2, \ldots, j_N) \in \{0, 1, 2, 3\}^N. \)
- \( X_{j} := \sigma_{(0, \ldots, 0, 1_{j^{th}} 0, \ldots, 0)}, \quad Y_{j} := \sigma_{(0, \ldots, 0, 2_{j^{th}} 0, \ldots, 0)}, \) and \( Z_{j} := \sigma_{(0, \ldots, 0, 3_{j^{th}} 0, \ldots, 0)}. \)
Pauli group

The $N$-qubit Pauli group $\mathcal{P}_N$ is defined as the sub-group of the unitary group $\mathcal{U}_n$ on $N$-qubits generated by the $X_j$, $Y_j$ and $Z_j$, $j = 1, \ldots, N$, as follows

$$\mathcal{P}_N := \langle X_j, Y_j, Z_j : 1 \leq j \leq N \rangle.$$
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$$\mathcal{P}_N := \{ \alpha \sigma_{(j_1, j_2, \ldots, j_N)} : j_l \in \{0, 1, 2, 3\}, \alpha \in \{ \pm 1, \pm \iota \} \}$$
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Pauli channel

A quantum channel is called a Pauli channel if each of its error operators are scalar multiple of elements in Pauli group \(\mathcal{P}_N\).
Correctable quantum channel

Quantum error correction code

A $k(= 2^p)$-dimensional subspace $V$ of $\mathbb{C}^n$ is called a quantum error correction code (QECC) for $\mathcal{E}$. 
Correctable quantum channel

Quantum error correction code

A $k(=2^p)$-dimensional subspace $\mathbb{V}$ of $\mathbb{C}^n$ is called a quantum error correction code (QECC) for $\mathcal{E}$ if there exists a quantum channel $\mathcal{R} : M_n \to M_n$ such that

$$\mathcal{R} \circ \mathcal{E}(\rho) = \rho \quad \text{for all} \quad P_\mathbb{V} \rho P_\mathbb{V} = \rho,$$

where $P_\mathbb{V}$ is an orthogonal projection onto the subspace $\mathbb{V}$. 
Operator approach to quantum error correction

If one write $P_V = U(I_k \oplus 0)U^\dagger$ for some unitary $U$, then

$$P_V \rho P_V = \rho \iff \rho = U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger.$$
Operator approach to quantum error correction

- If one write $P_{V} = U(I_{k} \oplus 0)U^{\dagger}$ for some unitary $U$, then

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- The equation of recovery channel can be restated as

$$\mathcal{R} \circ \mathcal{E}(U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^{\dagger}) = U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^{\dagger} \text{ for all } \hat{\rho} \in M_{k}.$$
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Operator approach to quantum error correction

- If one write $P_V = U(I_k \oplus 0)U^\dagger$ for some unitary $U$, then
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P_V \rho P_V = \rho \iff \rho = U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger.
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- The equation of recovery channel can be restated as
  \[
  \mathcal{R} \circ \mathcal{E} \left( U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger \right) = U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger \quad \text{for all} \quad \hat{\rho} \in M_k.
  \]

$\hat{\rho} \rightarrow \rho = U(\ket{0}\bra{0} \otimes \hat{\rho}) U^\dagger$

$p$-qubit data

Encoding to $N$-qubit
Operator approach to quantum error correction

- If one write $P_V = U(I_k \oplus 0)U^\dagger$ for some unitary $U$, then

$$P_V \rho P_V = \rho \iff \rho = U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger.$$  

- The equation of recovery channel can be restated as

$$\mathcal{R} \circ \mathcal{E}(U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger) = U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger \text{ for all } \hat{\rho} \in M_k.$$

\[\hat{\rho} \xrightarrow{\rho = U(|0\rangle \langle 0| \otimes \hat{\rho})U^\dagger} \mathcal{E}\]
Operator approach to quantum error correction

- If one write $P_V = U(I_k \oplus 0)U^\dagger$ for some unitary $U$, then

$$P_V \rho P_V = \rho \iff \rho = U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger.$$ 

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Operator approach to quantum error correction

- If one write $P_{\mathcal{V}} = U(I_k \oplus 0)U^\dagger$ for some unitary $U$, then

\[ P_{\mathcal{V}} \rho P_{\mathcal{V}} = \rho \iff \rho = U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger. \]

- The equation of recovery channel can be restated as

\[ \mathcal{R} \circ \mathcal{E}(U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger) = U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger \text{ for all } \hat{\rho} \in M_k. \]

$\rho$-qubit data $\xrightarrow{\text{Encoding to } N}$-qubit $\xrightarrow{\text{Noisy channel}}$ $\xrightarrow{\text{Recovery channel}}$ $\xrightarrow{\text{Decoding to } \rho}$-qubit

\[ \rho = U(|0\rangle\langle 0| \otimes \hat{\rho})U^\dagger \]

\[ \mathcal{R} \circ \mathcal{E}(U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger) = U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger \text{ for all } \hat{\rho} \in M_k. \]
Operator approach to quantum error correction

- If one write $P_V = U(I_k \oplus 0)U^\dagger$ for some unitary $U$, then

$$P_V \rho P_V = \rho \iff \rho = U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger.$$ 

- The equation of recovery channel can be restated as

$$\mathcal{R} \circ \mathcal{E}(U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger) = U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger \text{ for all } \hat{\rho} \in M_k.$$ 

\[\hat{\rho} \rightarrow \rho = U(\langle 0 | \otimes \hat{\rho})U^\dagger \rightarrow \mathcal{E} \rightarrow \mathcal{R} \rightarrow \text{tr}_1(U^\dagger \rho U) \rightarrow \hat{\rho}\]

$p$-qubit data

Encoding to $N$-qubit

Noisy channel

Recovery channel

Decoding to $p$-qubit
If one write $P_V = U(I_k \oplus 0)U^\dagger$ for some unitary $U$, then

$$P_V \rho P_V = \rho \iff \rho = U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger.$$ 

The equation of recovery channel can be restated as

$$\mathcal{R} \circ \mathcal{E}(U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger) = U \begin{bmatrix} \hat{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger \text{ for all } \hat{\rho} \in M_k.$$ 

When will such quantum error correction code exist??
Knill-Laflamme condition

Existence of QECC [Knill, Laflamme (1996)]

A quantum channel $\mathcal{E}(\rho) = \sum_{j=1}^{r} E_j \rho E_j^\dagger$ is correctable if and only if there exist $\gamma_{ij}$ such that

$$P_V E_i^\dagger E_j P_V = \gamma_{ij} P_V$$

for all $1 \leq i, j \leq r$.  
(QEC condition)
Knill-Laflamme condition

Existence of QECC [Knill, Laflamme (1996)]

A quantum channel $\mathcal{E}(\rho) = \sum_{j=1}^{r} E_j \rho E_j^\dagger$ is correctable if and only if there exist $\gamma_{ij}$ such that

$$P_V E_i^\dagger E_j P_V = \gamma_{ij} P_V$$

for all $1 \leq i, j \leq r$. (QEC condition)


Definition (rank-$k$ numerical range)

Let $A \in M_n(\mathbb{C})$ and let $k \geq 1$.

$$\Lambda_k(A) = \{ \lambda \in \mathbb{C} : W^\dagger A W = \lambda I_k \text{ with } W^\dagger W = I_k \}.$$
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Joint rank-$k$ numerical range

$$A = (A_1, \ldots, A_m) \in M_n^m$$

$$a = (a_1, \ldots, a_m) \in \mathbb{C}^{1 \times m}$$

$$\Lambda_k(A) = \{a \in \mathbb{C}^{1 \times m} : PAP = aP \}
\text{for some rank-}k \text{ orthogonal projection } P.$$
Joint rank-$k$ numerical range

\[ A = (A_1, \ldots, A_m) \in M_n^m \quad \text{and} \quad a = (a_1, \ldots, a_m) \in \mathbb{C}^{1 \times m} \]

\[ \Lambda_k(A) = \{ a \in \mathbb{C}^{1 \times m} : PAP = aP \} \]

for some rank-$k$ orthogonal projection $P$.

- A channel $\mathcal{E}$ has a $k$-dimensional QECC if and only if

  \[ \Lambda_k(E_1^\dagger E_1, E_1^\dagger E_2, \ldots, E_1^\dagger E_r, E_2^\dagger E_1, \ldots, E_r^\dagger E_r) \neq \emptyset. \]
Joint rank-$k$ numerical range

$A = (A_1, \ldots, A_m) \in M_n^m$ \hspace{1cm} $a = (a_1, \ldots, a_m) \in \mathbb{C}^{1 \times m}$

$\Lambda_k(A) = \{a \in \mathbb{C}^{1 \times m} : PAP = aP \}
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- A channel $\mathcal{E}$ has a $k$-dimensional QECC if and only if

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- Equivalently,

\[ \Lambda_k(A) = \{a \in \mathbb{C}^{1 \times m} : W^\dagger AW = aI_k \text{ with } W^\dagger W = I_k \}. \]
Joint rank-$k$ numerical range

\[ \mathbf{A} = (A_1, \ldots, A_m) \in M_n^m \quad \mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{C}^{1 \times m} \]

\[ \Lambda_k(\mathbf{A}) = \{ \mathbf{a} \in \mathbb{C}^{1 \times m} : P\mathbf{A}P = \mathbf{a}P \text{ for some rank}-k \text{ orthogonal projection } P \} \].

- A channel $\mathcal{E}$ has a $k$-dimensional QECC if and only if
  \[ \Lambda_k(E_1^\dagger E_1, E_1^\dagger E_2, \ldots, E_1^\dagger E_r, E_2^\dagger E_1, \ldots, E_r^\dagger E_r) \neq \emptyset. \]
- Equivalently,
  \[ \Lambda_k(\mathbf{A}) = \{ \mathbf{a} \in \mathbb{C}^{1 \times m} : W^\dagger \mathbf{A} W = \mathbf{a}I_k \text{ with } W^\dagger W = I_k \} \].
- Also, $\mathbf{a} \in \Lambda_k(\mathbf{A})$ if and only if there is a unitary $U$ such that
  \[ U^\dagger \mathbf{A} U = \begin{bmatrix} \mathbf{a}I_k & * \\ * & * \end{bmatrix}. \]
Joint rank-$k$ numerical range

- Write $A_j = H_{2j-1} + \nu H_{2j}$ with Hermitian matrices

\[ H_{2j-1} = \frac{1}{2}(A_j + A_j^\dagger) \quad \text{and} \quad H_{2j} = \frac{1}{2\nu}(A_j - A_j^\dagger). \]

- One can always identify

\[ \Lambda_k(A_1, \ldots, A_m) \cong \Lambda_k(H_1, H_2, \ldots, H_{2m-1}, H_{2m}) \]

\[ \subseteq \mathbb{C}^{1 \times m} \quad \text{and} \quad \subseteq \mathbb{R}^{1 \times 2m} \]

- One can focus on $\Lambda_k(A_1, \ldots, A_m)$ with $A_1, \ldots, A_m$ Hermitian.

- In particular, $\Lambda_k(A_1 + \nu A_2) \cong \Lambda_k(A_1, A_2)$. 
**Definition**

For $A = (A_1, \ldots, A_m) \in \mathcal{H}^m_n$, define

$$
\Omega_k(A) = \left\{ a \in \mathbb{R}^{1 \times m} : c \cdot a \leq \lambda_k(c \cdot A) \text{ for all unit vector } c \in \mathbb{R}^{1 \times m} \right\},
$$

where $\lambda_k(H)$ denotes the $k$-th largest eigenvalue of the Hermitian matrix $H$. 
**Definition**

For $A = (A_1, \ldots, A_m) \in \mathcal{H}_n^m$, define

$$\Omega_k(A) = \left\{ a \in \mathbb{R}^{1 \times m} : c \cdot a \leq \lambda_k(c \cdot A) \text{ for all unit vector } c \in \mathbb{R}^{1 \times m} \right\},$$

where $\lambda_k(H)$ denotes the $k$-th largest eigenvalue of the Hermitian matrix $H$.

**Observations**

- $\Lambda_k(A) \subseteq \Omega_k(A)$;
- $\Lambda_k(A) = \Omega_k(A)$, when $m = 1$; [Choi, Kribs, Zyczkowski (2006)]
- $\Lambda_k(A) = \Omega_k(A)$, when $m = 2$; [Li and Sze (2008)]
JHRNR of a commutative family

Let \{ A_1, \ldots, A_m \} be a commuting family in \( \mathcal{H}_n \).

Definition

The joint spectrum of Hermitian \( m \)-tuple \( A \in \mathcal{H}_n^m \) is defined as

\[
\text{spec}(A) := \{ \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^{1 \times m} : \exists 0 \neq x \in \mathbb{C}^n \text{ s.t. } Ax = \lambda x \}.
\]
Let \( \{ A_1, \ldots, A_m \} \) be a commuting family in \( \mathcal{H}_n \).

**Definition**

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\]

**Definition**

Let \( \text{spec}(A) = \{ \lambda_1, \ldots, \lambda_n \} \). Define

\[
\Delta_k(A) := \{ a \in \mathbb{R}^{1 \times m} : a \in \text{conv} (\{ \lambda_j : j \in S_i \}) \text{ for some disjoint subsets } S_1, S_2, \ldots, S_k \subseteq \{ 1, \ldots, n \} \}
\]
JHRNR of a commuting family

Proposition

Let \( \{ A_1, \ldots, A_m \} \) be a commuting family in \( \mathcal{H}_n \), and let \( A = (A_1, \ldots, A_m) \), with \( \text{spec}(A) = \{ \lambda_1, \ldots, \lambda_n \} \). Then

\[
\Delta_k(A) \subseteq \Lambda_k(A) \subseteq \Omega_k(A) = \bigcap_{\Gamma \subseteq \{1, \ldots, n\}, |\Gamma| = n-k+1} \text{conv} \{ \lambda_j : j \in \Gamma \}.
\]
JHRNR of a commuting family

**Proposition**

Let \( \{ A_1, \ldots, A_m \} \) be a commuting family in \( \mathcal{H}_n \), and let \( A = (A_1, \ldots, A_m) \), with \( \text{spec}(A) = \{ \lambda_1, \ldots, \lambda_n \} \). Then

\[
\Delta_k(A) \subseteq \Lambda_k(A) \subseteq \Omega_k(A) = \bigcap_{\Gamma \subseteq \{1, \ldots, n\}, \ |\Gamma| = n-k+1} \text{conv} \{ \lambda_j : j \in \Gamma \}.
\]

**Fact**

Let \( \{ A_1, \ldots, A_m \} \) be a commuting family in \( \mathcal{H}_n \) and let \( A = (A_1, \ldots, A_m) \). Then

\[
\Lambda_1(A) = \Omega_1(A) = \text{conv}(\text{spec}(A)).
\]
Let \{ A_1, \ldots, A_m \} \subseteq \mathcal{H}_n and let \( A = (A_1, \ldots, A_m) \). We know that
\[
\Lambda_k(A) \subseteq \Lambda_{kk'}(A \otimes I_{k'}). 
\]

**Question**

\[
\Lambda_{kk'}(A \otimes I_{k'}) \supseteq \Lambda_k(A). 
\]

Note that \( \Omega_{kk'}(A \otimes I_{k'}) = \Omega_k(A) \).
Let \( \{ A_1, \ldots, A_m \} \subseteq \mathcal{H}_n \) and let \( A = (A_1, \ldots, A_m) \). We know that

\[
\Lambda_k(A) \subseteq \Lambda_{kk'}(A \otimes I_{k'}).
\]

**Question**

\[
\Lambda_{kk'}(A \otimes I_{k'}) \subseteq \Lambda_k(A) ?
\]

Note that \( \Omega_{kk'}(A \otimes I_{k'}) = \Omega_k(A) \).
Notations

- As the phase factors $\mathcal{I} = \{ \pm 1, \pm \iota \}$ do not affect emptiness and non-emptiness of joint higher-rank numerical range of some elements in $\mathcal{P}_N$, we define an equivalence relation on $\mathcal{P}_N$ as follows:

  $$g \sim h \iff g = \alpha h \quad \text{for some } \alpha \in \mathcal{I}.$$  

- Suppose $[g] = \{ \alpha g : \alpha \in \mathcal{I} \}$ is equivalence class of an operator $g \in \mathcal{P}_N$. We suppose $[\alpha \sigma_{(j_1, \ldots, j_N)}] = \sigma_{(j_1, \ldots, j_N)}, \alpha \in \mathcal{I}$. This means that $\sigma_{(j_1, \ldots, j_N)}$ is the representative of the class $[\alpha \sigma_{(j_1, \ldots, j_N)}]$.

- Let $S$ be a subset of $\mathcal{P}_N$. We indicate the set of all representatives of equivalence classes in $S$ as $[S] = \{ [g] : g \in S \}$. So

  $$[\mathcal{P}_N] = \{ \sigma_{(j_1, \ldots, j_N)} : (j_1, \ldots, j_N) \in \{ 0, 1, 2, 3 \}^N \}.$$  

Let \( \{ A_1, A_2, A_3 \} \subseteq [\mathcal{P}_N] \) be an independent set, such that none of its members is \( I_{2^N} \) and let \( A = (A_1, A_2, A_3) \).

- If \( A_i A_j = -A_j A_i, i \neq j \), then for every \( 1 \leq k \leq 2^{N-1} \) we have
  - If \( A_1 A_2 \) and \( A_3 \) are dependent and \( 2^{N-2} < k \leq 2^{N-1} \), then
    
    \[
    \text{conv} (\Lambda_k(A)) = \Omega_k(A) = \{ a \in \mathbb{R}^{1 \times 3} : \|a\| \leq 1 \};
    \]

- Otherwise,
  
  \[
  \Lambda_k(A) = \Omega_k(A) = \{ a \in \mathbb{R}^{1 \times 3} : \|a\| \leq 1 \}. \]
Higher-rank Numerical Range and Quantum Error Correction
Joint Higher-rank Numerical Range

**JHRNR of three elements in** $P_N$

- If $A_1 A_2 = -A_2 A_1$ and $A_1 A_3 = -A_3 A_1$, but $A_2 A_3 = A_3 A_2$, then
  - for every $2^{N-2} < k \leq 2^{N-1}$ we have
    \[
    \Lambda_k(A) = \Omega_k(A) = \{ (a_1, 0, 0) \in \mathbb{R}^{1 \times 3} : |a_1| \leq 1 \};
    \]
- For every $1 \leq k \leq 2^{N-2}$ we have
  \[
  \Lambda_k(A) = \Omega_k(A) = \{ (a_1, a_2, a_3) \in \mathbb{R}^{1 \times 3} : |a_1| \leq 1, |a_2| \leq \sqrt{1 - a_1^2}, |a_3| \leq \sqrt{1 - a_1^2} \};
  \]
If $A_1 A_2 = A_2 A_1$, and $A_1 A_3 = A_3 A_1$, but $A_2 A_3 = -A_3 A_2$, then

- for every $2^{N-2} < k \leq 2^{N-1}$ we have

$$\Lambda_k(A) = \Omega_k(A) = \{(0, 0, 0)\};$$

- for every $1 \leq k \leq 2^{N-2}$ we have

$$\Lambda_k(A) = \Omega_k(A)$$

$$= \{(a_1, a_2, a_3) \in \mathbb{R}^{1 \times 3} : |a_1| \leq 1, |a_2|^2 + |a_3|^2 \leq 1\}.$$
JHRNR of three elements in $\mathcal{P}_N$

- Let $A_iA_j = A_jA_i$.
  - If $A_1A_2$ and $A_3$ are dependent, then
    
    $\Lambda_k(A) = \Omega_k(A) = \emptyset$, for every $2^{N-2} < k \leq 2^{N-1}$;
    
    $\Lambda_k(A) = \Omega_k(A) = \text{conv}(\text{spec}(A))$, for every $1 \leq k \leq 2^{N-2}$;
  
  - If $A_1A_2$ and $A_3$ are independent, then
    
    $\Lambda_k(A) = \Omega_k(A) = \{(0,0,0)\}$, for every $2^{N-2} < k \leq 2^{N-1}$;
    
    $\Lambda_k(A) = \Omega_k(A) = \text{conv}\{(\pm 1,0,0), (0,\pm 1,0), (0,0,\pm 1)\}$, for every $2^{N-3} < k \leq 2^{N-2}$;
    
    $\Lambda_k(A) = \Omega_k(A) = \text{conv}(\text{spec}(A))$, for every $1 \leq k \leq 2^{N-3}$. 

Let \( \{ A_1, A_2, A_3 \} \subseteq [\mathcal{P}_N] \) be an independent set, such that none of its members is \( I_{2^N} \) and let \( A = (A_1, A_2, A_3) \).

1. If \( A_1 A_2 \) and \( A_3 \) are dependent, \( A_1 A_2 = -A_2 A_1 \), and \( 2^{N-2} < k \leq 2^{N-1} \), then \( \text{conv} (\Lambda_k(A)) = \Omega_k(A) \);

2. Otherwise, \( \Lambda_k(A) = \Omega_k(A) \).
Caracterization of JHRNR

Let \( \{ A_1, \ldots, A_m \} \subseteq \mathcal{H}_n \) and let \( A = (A_1, \ldots, A_m) \). We know that

\[
\text{conv} \left( \Lambda_k(A) \right) \subseteq \Omega_k(A).
\]

**Question**

\[
\Omega_k(A) \subseteq \text{conv} \left( \Lambda_k(A) \right).
\]
Caracterization of JHRNR

Let \( \{ A_1, \ldots, A_m \} \subseteq \mathcal{H}_n \) and let \( A = (A_1, \ldots, A_m) \). We know that

\[
\text{conv} (\Lambda_k(A)) \subseteq \Omega_k(A).
\]

Question

\[
\Omega_k(A) \subseteq \text{conv} (\Lambda_k(A)).
\]

Counter example

If \( A_j = \sigma_j \oplus [0], j = 1, 2, 3 \), then \( \Omega_2(A_1, A_2, A_3) = \{ (0, 0, 0) \} \) but \( \Lambda_2(A_1, A_2, A_3) = \emptyset \).
Caracterization of JHRNR

Let \( \{ A_1, \ldots, A_m \} \subseteq [\mathcal{P}_N] \) and let \( A = (A_1, \ldots, A_m) \).

Question

\[ \Omega_k(A) \subseteq \text{conv} (\Lambda_k(A)) . \]

Question

\[ \Lambda_k(A) \neq \emptyset \implies 0 \in \Lambda_k(A) \]
\[ \Omega_k(A) \neq \emptyset \implies 0 \in \Omega_k(A) \]
Maximal abelian sub-group

Let $G$ be an arbitrary sub-group of $\mathcal{P}_N$, which is not a set of scalar matrices.

- A maximal abelian sub-group $S$ of $G$ is an abelian sub-group of $G$ such that no abelian sub-group of $G$ contains $S$ strictly.
Maximal abelian sub-group

Let $G$ be an arbitrary sub-group of $\mathcal{P}_N$, which is not a set of scalar matrices.

- A maximal abelian sub-group $S$ of $G$ is an abelian sub-group of $G$ such that no abelian sub-group of $G$ contains $S$ strictly.

- We suppose that a minimal generating set $G_0$ of $G$ do not contain any scalar matrices; Because if $\alpha I_{2N}, \beta \sigma_{(j_1, \ldots, j_N)} \in G_0$ for some $\alpha, \beta \in \{ \pm 1, \pm i \}$ and $(j_1, \ldots, j_N) \in \{ 0, 1, 2, 3 \}^N$, where $(j_1, \ldots, j_N) \neq (0, \ldots, 0)$, then we can replace $\alpha I_{2N}$ with $\alpha \overline{\beta} \sigma_{(j_1, \ldots, j_N)}$. 
Maximal abelian sub-group

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- Let $|G| = m$. Then $G$ represents an $m$-tuple of elements in the set $[G]$ with an arbitrary but fixed order.
Maximal abelian sub-group

Lemma

Let $G$ be a sub-group of $\mathcal{P}_N$ and let $S_0$ be a minimal generating set for a maximal abelian sub-group $S$ of $G$, where $[S_0] = \{ g_1, \ldots, g_m \}$. Then we can find a unitary $U$ in the Clifford group such that $U^\dagger g_j U = Z_j$, $1 \leq j \leq m$, where $U \in N_{U_n}(\mathcal{P}_N)$. 
Maximal abelian sub-group

**Lemma**

Let $G$ be a sub-group of $\mathcal{P}_N$ and let $S_0$ be a minimal generating set for a maximal abelian sub-group $S$ of $G$, where $[S_0] = \{ g_1, \ldots, g_m \}$. Then we can find a unitary $U$ in the Clifford group such that $U^\dagger g_j U = Z_j$, $1 \leq j \leq m$, where $U \in N_{\mathcal{U}_n}(\mathcal{P}_N)$.

$$N_{\mathcal{U}_n}(\mathcal{P}_N) = \{ U \in \mathcal{U}_n : U^\dagger \mathcal{P}_N U = \mathcal{P}_N \}.$$
JHRNR for a sub-group of N-qubit Pauli group

**Theorem**

Let $G$ be an abelian sub-group of the $N$-qubit Pauli group $\mathcal{P}_N$ with a minimal generating set $G_0$, where $|[G_0]| = m$. Then $\Lambda_{2N-m+1}(G) = \emptyset$ and $\Lambda_{2N-m}(G) = \text{conv} (\text{spec}(G))$.

Note that, if $[G_0] = \{Z_1, \ldots, Z_m\}$, then we obtain that $W^\dagger GW = (1, 1, \ldots, 1)I_{2N-m}$, where

$$W = \begin{bmatrix} I_{2N-m} & 0_{2N-m} & \cdots & 0_{2N-m} \end{bmatrix}^\dagger.$$
Theorem

Let $G$ be a non-abelian sub-group of $\mathcal{P}_N$. Let $S$ be a maximal abelian sub-group of $G$ with a minimal generating set $S_0$, where $|[S_0]| = m$. Then $\Lambda_{2N-m+1}(G) = \emptyset$ and $\Lambda_{2N-m}(G) \neq \emptyset$.

Note that, if $[S_0] = \{Z_1, \ldots, Z_m\}$, then we obtain that $W^\dagger gW = I_{2N-m}$, for every $g \in [S]$ and $W^\dagger gW = 0_{2N-m}$, for every $g \in [G] \setminus [S]$, where

$$W = \begin{bmatrix} I_{2N-m} & 0_{2N-m} & \cdots & 0_{2N-m} \end{bmatrix}^\dagger.$$
JHRNR for a sub-group of $N$-qubit Pauli group

Example

Let $N > 2$ and let $G$ be a subgroup of $\mathcal{P}_N$ with minimal generating set $G_0$. Suppose $[G_0] = \{ \sigma_3^\otimes N, \sigma_1^\otimes N \}$, where $\sigma_i^\otimes N := \sigma_i \otimes \sigma_i \otimes \cdots \otimes \sigma_i$. So

$$[G] = \{ I_{2N}, \sigma_1^\otimes N, \sigma_2^\otimes N, \sigma_3^\otimes N \}.$$ 

- If $N$ is even, then $G$ is an abelian sub-group of $\mathcal{P}_N$ and so $\Lambda_{2N-2}(G) \neq \emptyset$.
- When $N$ is odd, then $G$ is a non-abelian sub-group of $\mathcal{P}_N$ and we can choose a minimal generating set $S_0$ for a maximal abelian sub-group of $G$ with $[S_0] = \{ \sigma_3^\otimes N \}$. Thus $\Lambda_{2N-1}(G) \neq \emptyset$. 
Contents

1. Operator Approach to Quantum Error Correction
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3. Quantum Error Correction of Pauli Channels
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QEC for an abelian sub-group of $\mathcal{P}_N$

**Proposition**

Let $G$ be an abelian sub-group of the $N$-qubit Pauli group $\mathcal{P}_N$ with a minimal generating set $G_0$, where $N > |[G_0]| = m$. Let $\mathcal{E}$ be a Pauli channel, where its noise operators are members of $[G]$. Then there exists a unitary matrix $U \in \mathcal{U}_n$ such that

$$U^\dagger \mathcal{E} \left( U \left( |0\rangle \langle 0| \otimes \hat{\rho} \right) U^\dagger \right) U = |0\rangle \langle 0| \otimes \hat{\rho},$$

for all $\hat{\rho} \in M_{2^{N-m}}$.

Moreover, there is no $2^{N-m+1}$-dimensional QECC for $\mathcal{E}$ and in this sense QEC is optimal.
QEC for an abelian sub-group of $\mathcal{P}_N$

Note that, if $[G_0] = \{Z_1, \ldots, Z_m\}$, then we have the following

$$\mathcal{E}(\langle 0 | \otimes \hat{\rho} ) = \langle 0 | \otimes \hat{\rho} \quad \text{for all } \hat{\rho} \in M_{2^{N-m}},$$

that is a decoherence free subspace QEC.
QEC for a non-abelian sub-group of $\mathcal{P}_N$

**Proposition**

Let $G$ be a non-abelian sub-group of $\mathcal{P}_N$ and let $S$ be a maximal abelian sub-group of $G$ with a minimal generating set $S_0$, where $N > |[S_0]| = m$. Let $\mathcal{E}$ be a Pauli channel, where its noise operators are members of $[G]$. Then there are a density matrix $\sigma \in \mathbb{C}^{2^m \times 2^m}$ and a unitary matrix $R \in \mathcal{U}_n$ such that

$$R^\dagger \mathcal{E} \left( U(|0\rangle \langle 0| \otimes \hat{\rho}) U^\dagger \right) R = \sigma \otimes \hat{\rho}, \quad \text{for all } \hat{\rho} \in M_{2^{N-m}}.$$ 

Moreover, this QEC is optimal as there is no $(N - m + 1)$-dimensional QECC for $\mathcal{E}$. 
QEC for a non-abelian sub-group of $\mathcal{P}_N$

If $[S_0] = \{ Z_1, \ldots, Z_m \}$, then $U = I_{2N}$ and $R = \text{diag}(B_0, B_1, \ldots, B_{2^m-1})$. Define $V_1 = \text{diag}(B_0, B_1, \ldots, B_{2^m-1-1})$ and $V_2 = \text{diag}(B_{2^m-1}, \ldots, B_{2^m-1})$. Note that in this case $R$ is product of controlled[0]-$V_1$ and controlled[1]-$V_2$ quantum gates.

Moreover, if for every $h \in [G]$, $h = \sigma_{(j_1, \ldots, j_m)} \otimes I_{2N-m}$, then

$$\mathcal{E}(|0\rangle \langle 0| \otimes \hat{\rho}) = \sigma \otimes \hat{\rho} \quad \text{for all} \quad \hat{\rho} \in M_{2^{N-m}},$$

that is a decoherence free sub-system QEC.
QEC for a non-abelian sub-group of $\mathcal{P}_N$
QEC for a non-abelian sub-group of $\mathcal{P}_N$

Let $G$ be a sub-group of $\mathcal{P}_N$ with minimal generating set

$$G_0 = \{ Z_1, Z_2, X_2, X_1X_3 \}.$$ 

We can choose a minimal generating set $S_0$ for maximal abelian sub-group $S$ of $G$ such that $[S_0] = \{ Z_1, Z_2 \}$. So $[G] = \{ g_1, \ldots, g_{16} \}$, where

$$
\begin{align*}
g_1 &= \sigma_{(0,0,0)} \otimes I_{2^{N-3}}, & g_2 &= \sigma_{(3,0,0)} \otimes I_{2^{N-3}}, & g_3 &= \sigma_{(0,3,0)} \otimes I_{2^{N-3}}, & g_4 &= \sigma_{(3,3,0)} \otimes I_{2^{N-3}}, \\
g_5 &= \sigma_{(0,1,0)} \otimes I_{2^{N-3}}, & g_6 &= \sigma_{(3,1,0)} \otimes I_{2^{N-3}}, & g_7 &= \sigma_{(0,2,0)} \otimes I_{2^{N-3}}, & g_8 &= \sigma_{(3,2,0)} \otimes I_{2^{N-3}}, \\
g_9 &= \sigma_{(1,0,1)} \otimes I_{2^{N-3}}, & g_{10} &= \sigma_{(2,0,1)} \otimes I_{2^{N-3}}, & g_{11} &= \sigma_{(1,3,1)} \otimes I_{2^{N-3}}, & g_{12} &= \sigma_{(2,3,1)} \otimes I_{2^{N-3}}, \\
g_{13} &= \sigma_{(1,1,1)} \otimes I_{2^{N-3}}, & g_{14} &= \sigma_{(2,1,1)} \otimes I_{2^{N-3}}, & g_{15} &= \sigma_{(1,2,1)} \otimes I_{2^{N-3}}, & g_{16} &= \sigma_{(2,2,1)} \otimes I_{2^{N-3}}.
\end{align*}
$$
QEC for a non-abelian sub-group of $\mathcal{P}_N$

Let

$$\mathcal{E}(\rho) = \sum_{i=1}^{16} p_i g_i \rho g_i^\dagger, \quad \sum_{i=1}^{16} p_i = 1,$$

be a quantum channel, then we obtain that

$$R^\dagger \mathcal{E} (|0\rangle \langle 0| \otimes \hat{\rho}) R = \text{diag}(d_1, d_2, d_3, d_4) \otimes \hat{\rho}, \quad \text{for all } \hat{\rho} \in \mathbb{C}^{2^{N-2} \times 2^{N-2}},$$

where $R = \text{diag}(\sigma_0, \sigma_0, \sigma_1, \sigma_1) \otimes I_{2^{N-3}}$, $d_1 = p_1 + \cdots + p_4$, $d_2 = p_1 + \cdots + p_8$, $d_3 = p_9 + \cdots + p_{12}$, and $d_4 = p_{13} + \cdots + p_{16}$. 
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Thanks for your attention.
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