

A Matrix Decomposition and Its Applications

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With help and input from M.H. Lin

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WONRA14 Organizers

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- Man-Duen Choi, University of Toronto
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- Jie Xiao, Tsinghua University



with Xiao Jie at Beijing Normal U. 1989



Who were there? *Town Musicians of Bremen*



Who were there?

Why Matrix Decompositions/Factorizations

Linear equation system:

$$Ax = b$$

↓

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & * \end{pmatrix} x = b'.$$

Row echelon form - a special matrix form

Matrix Decompositions/factorizations

- Schur triangular decomposition: $A = U^*TU$, where U is unitary, T is upper-triangular.
- Polar decomposition: $A = PU$, where P is positive semidefinite, U is unitary.
- Singular value decomposition: $A = SDV$, where D is diagonal with the singular values of A , S and V are unitary.
- Spectral decomposition for normal matrices: $A = U^*DU$, where D is diagonal and U is unitary.
- LU decomposition for certain matrices: $A = LU$, where L is lower-triangular and U is upper-triangular.
- QR decomposition: $A = QR$, where Q has orthogonal columns and R is upper-triangular.
- more ... (See Horn & Johnson *Matrix Analysis*, 2nd ed. CUP 2013) or Zhang *Matrix Theory*, Springer 2011)

How should we define an angle between two vectors



Dresses: Beach vs. Wedding

A special matrix decomposition

Theorem (Sectoral decomposition)

Let A be an $n \times n$ complex matrix such that $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then there exist an invertible matrix X and a unitary diagonal matrix $Z = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$ with all $|\theta_j| \leq \alpha$ such that

$$A = XZX^*.$$

Moreover, such a matrix Z is unique up to permutation.

Why this decomposition: Motivation I

Growth factor in Computation:

Consider the system of n linear equations of n unknowns

$$Ax = b$$

(after k steps of Gaussian elimination, $1 \leq k \leq n - 1$)

$$\begin{pmatrix} * & * \\ 0 & A^{(k)} \end{pmatrix} x = b'.$$

$$\rho(A) = \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}|}$$

$\rho(A)$ is called the *growth factor* of A in Gaussian Elimination.
Estimating $\rho(A)$ (G. Golub & C. Van Loan, G.W. Stewart, ...)

N. Higham's 1998 paper in Math. Comp.

LU factorization without pivoting:

- diagonally dominant;
- positive definite;
- totally nonnegative.

Higham shows

- If $A = B + iC$, where B, C are *real* positive definite (CSPD), then A admits LU decomposition without pivoting;
- The growth factor $\rho(A) \leq 2$.

Nicholas J. Higham:

Fellow of the Royal Society

SIAM Fellow

(<http://www.maths.manchester.ac.uk/higham/>)

Professor Nicholas J. Higham

Richardson Professor of Applied
Mathematics 
Fellow  of the Royal Society 
SIAM  Fellow 
ISI Highly Cited Researcher 

Citations

Google Scholar  profile
Aminer  profile
Microsoft Academic Search  profile
Researcher ID  profile and ORCID ID

Research



Contact details

School of Mathematics
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The University of Manchester

Books



N. Higham's 1998 paper, p.1594

Theorem 2.5. *The growth factor for a CSPD matrix $A \in \mathbb{C}^{n \times n}$ satisfies $\rho_n < 2$ and this bound is the best possible.*

Proof. In the notation of Lemma 2.3, we have to show that every element of the Schur complement matrix S satisfies $|s_{ij}| < 2 \max_{i,j} |a_{ij}|$. Since S is CSPD by Lemma 2.3, we have

$$\operatorname{Re}((A_{21}A_{11}^{-1}A_{21}^T)_{jj}) < \operatorname{Re}((A_{22})_{jj}), \quad \operatorname{Im}((A_{21}A_{11}^{-1}A_{21}^T)_{jj}) < \operatorname{Im}((A_{22})_{jj}).$$

Squaring, adding and then taking the square root gives

$$|(A_{21}A_{11}^{-1}A_{21}^T)_{jj}| < |(A_{22})_{jj}|.$$

Thus $|s_{jj}| < 2|(A_{22})_{jj}| < 2 \max_{i,j} |a_{ij}|$. Hence, by Lemma 2.4,

$$|s_{ij}| < \sqrt{|s_{ii}||s_{jj}|} < 2 \max_{i,j} |a_{ij}|.$$

Finally, consider the LU factorization

$$A = \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -i & 1 \end{bmatrix} \begin{bmatrix} 1+i & 1-i \\ 0 & 2(1+i) \end{bmatrix},$$

for which $\rho_2 = 2$. The matrix A is not CSPD, because its real and imaginary parts are symmetric positive semidefinite and singular. But $A + \epsilon(1+i)I$ is CSPD for $\epsilon > 0$ and as $\epsilon \rightarrow 0$, $\rho_2 \rightarrow 2$. Hence $\rho_n < 2$ is the best possible bound. $\square$

Higham's Conjecture?

Ikramov and Kucherov, 2000, in NLA,

$$\rho_n(A) = \frac{\max_{j,k} |a_{jj}^{(k)}|}{\max_j |a_{jj}|}$$

It was conjectured in Reference [1] (see Theorem 2.5) that

Higham's '98 paper

$$\rho_n(A) \leq 2$$

for a CSPD matrix A . We cannot prove this bound in the general case. He
a better bound than (5) for a particular subset of CSPD matrices examined
matrices

$$A = T + iG$$

Higham's Conjecture?

George, Ikramov and Kucherov, 2002, in NLA,

It was conjectured in Reference [2] that

$$\rho_n(A) \leq 2$$

for a Higham matrix A . The present authors were able to prove the better bound

$$\rho_n(A) \leq \frac{1 + \sqrt{17}}{4} \simeq 1.28078 \dots$$

Higham's Conjecture?

M. Lin, 2013, in Calcolo,

is called the growth factor (in Gaussian elimination) of A . For more information on the numerical significance of investigating growth factor and its connection with the stability of Gaussian elimination, we refer to [3] and references therein. It is proved in [3] that if A in (1) is a Higham matrix, then no pivoting is needed in Gaussian elimination. Higham [3] also conjectured that $\rho_n(A) \leq 2$ for such a matrix.

George et al. [2] made some progress concerning this conjecture. They obtained the following result:

there is no such
conjecture.

Theorem 1 *Let $A \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative. Then $\rho_n(A) < 3\sqrt{2}$. If A is a Higham matrix, then $\rho_n(A) < 3$.*

LI, GUO, AND GU, 2013 May,

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S 0025-5718(XX)0000-0

A NOTE ON THE GROWTH FACTOR IN GAUSSIAN ELIMINATION FOR GENERALIZED HIGHAM MATRICES

HOU-BIAO LI, QIAN-PING GUO, AND XIAN-MING GU

ABSTRACT. The Higham matrix is a complex symmetric matrix $A = B + iC$, where both B and C are real, symmetric and positive definite and $i = \sqrt{-1}$ is the imaginary unit. For any Higham matrix A , Ikramov et al showed that the growth factor in Gaussian elimination is less than 3. In this paper, based on the previous results, a new bound of the growth factor is obtained by using the maximum of the condition numbers of matrixes B and C for the generalized Higham matrix A , which strengthens this bound to 2 and proves the Higham's conjecture.

1. INTRODUCTION

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1. INTRODUCTION

Higham's Conjecture?

Higham's proof is incorrect (his 2002 SIAM book Accuracy and stability of numerical algorithms, 2nd ed., p.210), then it becomes a Research Problem (p.212).

of block LDL^T factorization with the Bunch-Kaufman pivoting strategy (see §11.1) when applied to such matrices. The proof of the growth factor bound in [610, 1998] is incorrect; the subsequent proof that $\rho_n < 3$ in [477, 2002] is quite lengthy, and obtaining a sharp bound for the growth factor is an open problem (see Problem 10.12).

Administrator

Higham's '98 paper

10.11. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{C}^{n \times n}$$

have positive definite Hermitian part. Show that the Schur complement $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ also has positive definite Hermitian part. In other words, show that GE preserves positive definiteness.

10.12. (RESEARCH PROBLEM) Investigate the sharpness of the bound $\rho_n < 3$ of [477, 2002] for matrices of the form (10.30).

N. Higham's 1998 paper, p.1594, what is wrong?

Theorem 2.5. *The growth factor for a CSPD matrix $A \in \mathbb{C}^{n \times n}$ satisfies $\rho_n < 2$ and this bound is the best possible.*

Proof. In the notation of Lemma 2.3, we have to show that every element of the Schur complement matrix S satisfies $|s_{ij}| < 2 \max_{i,j} |a_{ij}|$. Since S is CSPD by Lemma 2.3, we have

$$\operatorname{Re}((A_{21}A_{11}^{-1}A_{21}^T)_{jj}) < \operatorname{Re}((A_{22})_{jj}), \quad \operatorname{Im}((A_{21}A_{11}^{-1}A_{21}^T)_{jj}) < \operatorname{Im}((A_{22})_{jj}).$$

Squaring, adding and then taking the square root gives

$$|(A_{21}A_{11}^{-1}A_{21}^T)_{jj}| < |(A_{22})_{jj}|.$$

Thus $|s_{jj}| < 2|(A_{22})_{jj}| < 2 \max_{i,j} |a_{ij}|$. Hence, by Lemma 2.4,

$$|s_{ij}| < \sqrt{|s_{ii}||s_{jj}|} < 2 \max_{i,j} |a_{ij}|.$$

Finally, consider the LU factorization

$$A = \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -i & 1 \end{bmatrix} \begin{bmatrix} 1+i & 1-i \\ 0 & 2(1+i) \end{bmatrix},$$

for which $\rho_2 = 2$. The matrix A is not CSPD, because its real and imaginary parts are symmetric positive semidefinite and singular. But $A + \epsilon(1+i)I$ is CSPD for $\epsilon > 0$ and as $\epsilon \rightarrow 0$, $\rho_2 \rightarrow 2$. Hence $\rho_n < 2$ is the best possible bound. $\square$

Higham's Conjecture?

Emails: From: Ikramov To Higham, Tue, 30 Nov 1999

Dear Nick,

... I cannot understand your proof of Theorem 2.5. Why are you squaring an inequality like $\Re((A_{21}A_{11}^{-1}A_{21}^T)_{jj}) < \Re((A_{22})_{jj})$, whose left-hand side may be negative? Could you explain this to me? –Khakim

From Higham to Ikramov, Thu Dec 9, 1999

Dear Khakim,

You're right - there's an error in the proof of Thm 2.5.... I've not been able to spot a quick fix to the proof. But I feel that a short, elegant proof should be possible - if you think of one please let me know. Best wishes, —Nick

Higham's conjecture: $\rho \leq 2$

George, Ikramov, Kucherov, 2000, 2002, 2004, *Higham matrix=CSPD and Generalized Higham (accretive-dissipative) matrix*

$$A = B + iC = M + iN,$$

where B, C are real symmetric positive definite (rspd over $\mathbb{R}$)
 M, N are complex Hermitian positive definite (over $\mathbb{C}$).

If $A = B + iC$, where B, C are rspd, then $B = M, C = N$.

In general, $B \neq M, C \neq N$: Take $A = \begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix}$. Then

$$A = B + Ci = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = A + i0 = M + iN,$$

$$\rho \leq 4, 3, 2\sqrt{2}, 2.$$

S. Drury (2013LAA)

Lin's problem on the partitioned Higham's matrices:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{where } A_{11} \text{ and } A_{22} \text{ are square.}$$

$$|\det A| \leq \gamma |\det A_{11}| |\det A_{22}|.$$

S. Drury, M. Lin, C.-K. Li & N.-S. Sze ... "race"?

Speaking of race...

Positivity of Partitioned Hermitian Matrices with Unitarily Invariant Norms

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3301 College Ave, Fort Lauderdale, FL 33314, USA

In memory of Robert C. Thompson

Abstract

We give a short proof of a recent result of Drury on the positivity of a 3×3 matrix of the form $(\|R_i^* R_j\|_{\text{tr}})_{1 \leq i, j \leq 3}$ for any rectangular complex (or real) matrices R_1, R_2, R_3 so that the multiplication $R_i^* R_j$ is compatible for all i, j , where $\|\cdot\|_{\text{tr}}$ denotes the trace norm. We then give a complete analysis of the problem when the trace norm is replaced by other unitarily invariant norms.

CK's speed!

Extra line breaks in this message were removed.

From: CK Li <math.ckli@gmail.com>

Sent: Mon 6/16/2014 2:48 PM

To: Fuzhen Zhang

Cc:

Subject: Re: Office

Hi Fuzhen,

I have a short proof for the result of Drury. Will tell you when we meet.

Regards,

Chi Kwong

Why this decomposition: Motivation II

Let A be an n -square complex matrix partitioned in the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{where } A_{11} \text{ and } A_{22} \text{ are square.} \quad (1)$$

Lin & Zhou [J. Math. Anal. Appl. 2013]: Let $A \in \mathbb{M}_n$ be (accretive-dissipative and) partitioned as in (4). Then for any unitarily invariant norm $\|\cdot\|$ on $\mathbb{M}_n$,

$$\max\{\|A_{12}\|^2, \|A_{21}\|^2\} \leq 4\|A_{11}\| \|A_{22}\| \quad (2)$$

$$\|A\| \leq \sqrt{2} (\|A_{11}\| + \|A_{22}\|). \quad (3)$$

Recent developments by S. Drury, C.-K Li & R. Sze, M. Lin, etc.

Numerical range; sector

For $A \in \mathbb{M}_n$, the *numerical range* (or *field of values*) of A is the set in the complex plane

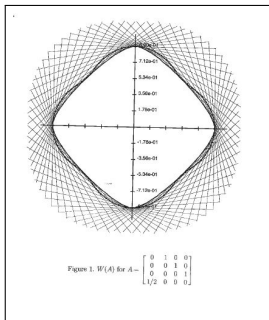
$$W(A) = \{x^*Ax \mid x \in \mathbb{C}^n, \|x\| = 1\}.$$

For $\alpha \in [0, \frac{\pi}{2})$, let S_α be the sector in the complex plane given by

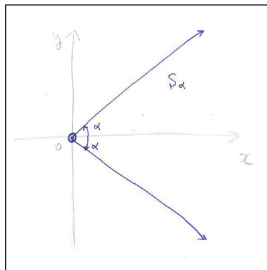
$$\begin{aligned} S_\alpha &= \{z \in \mathbb{C} \mid \operatorname{Re} z > 0, |\operatorname{Im} z| \leq \operatorname{Re} z \tan \alpha\} \\ &= \{re^{i\theta} \mid r > 0, |\theta| \leq \alpha\}. \end{aligned}$$

Stable matrix (or Hurwitz matrix): if every eigenvalue of has strictly negative real part, i.e. $\operatorname{Re}(\lambda_i(A)) < 0$.

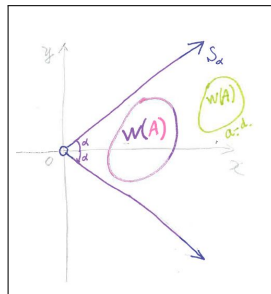
$W(A)$ and S_α



Gustafson & Rao,
Numerical Range,
Springer, 1997

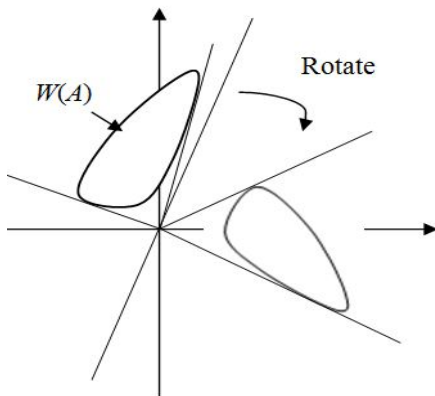


S_α : All points
between the two
rays.



$W(A)$ in S_α
 $W(A)$ in $S_{\pi/4}$
accretive-dissipative
matrix (a.d.m.)

Rotation



Numerical Range rotated

Theorem (Sectoral decomposition)

Let A be an $n \times n$ complex matrix such that $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then there exist an invertible matrix X and a unitary diagonal matrix $Z = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$ with all $|\theta_j| \leq \alpha$ such that

$$A = XZX^*.$$

Moreover, such a matrix Z is unique up to permutation.

Similar decomposition appeared in [D. London, AMS Proc. 1981].

The Cartesian (Toeplitz) decomposition

For a square complex matrix A , the Cartesian (or Toeplitz) decomposition is $A = \operatorname{Re} A + i \operatorname{Im} A = M + iN$, where

$$\operatorname{Re} A = \frac{1}{2}(A + A^*) = M, \quad \operatorname{Im} A = \frac{1}{2i}(A - A^*) = N.$$

M, N are Hermitian:

$$x^* A x = x^* M x + i(x^* N x).$$

If $M, N > 0$ (positive definite), then $W(A)$ is contained in the 1st quadrant; A is called *accretive-dissipative*. $W(e^{-\pi/4i} A) \subseteq S_{\pi/4}$.

Proof outline

Existence. $A = M + iN = \operatorname{Re} A + i \operatorname{Im} A$ (Cartesian decomposition). Since $W(A) \subseteq S_\alpha$, A is invertible and $M > 0$ (positive definite). M and N are simultaneously $*$ -congruent and diagonalizable [Z., Matrix Theory, p.213]. $A = QDQ^*$. Since $W(A) \subseteq S_\alpha$, $W(D) \subseteq S_\alpha$. Thus $D = \operatorname{diag}(d_1 e^{i\theta_1}, \dots, d_n e^{i\theta_n})$, where $d_j > 0$ and $|\theta_j| \leq \alpha$, $j = 1, \dots, n$. $A = XZX^*$, $X = Q \operatorname{diag}(\sqrt{d_1}, \dots, \sqrt{d_n})$ and $Z = \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$.

Uniqueness. $A = XZ_1X^* = YZ_2Y^*$. Assume $Y = I$. Show Z_1 and Z_2 have the same main diagonal entries. We show $\beta \in \mathbb{C}$ is a diagonal entry of Z_1 with multiplicity k if and only if β is a diagonal entry of Z_2 with the same multiplicity. WLOG, assume $\beta = 1$. Let $Z_1 = C_1 + iS_1$ and $Z_2 = C_2 + iS_2$. Since $\beta = 1$ is a diagonal entry of Z_1 with multiplicity k , 1 appears on the diagonal of C_1 k times, so S_1 has k zeros on its diagonal. Thus $\operatorname{rank}(XS_1X^*) = n - k$. As $XS_1X^* = S_2$, we have $\operatorname{rank}(S_2) = n - k$. So C_2 has k 1's on its diagonal, concluding Z_2 is permutation similar to Z_1 .

Corollary

Let A be an $n \times n$ complex matrix such that $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$ and let $A = XZX^$ be a sectoral decomposition of A , where X is invertible and Z is unitary and diagonal. Then*

- ❶ $I \leq \sec \alpha (\operatorname{Re} Z)$. (Loewner order)
- ❷ $RR^* \leq \sec \alpha (R(\operatorname{Re} Z)R^*)$ for any matrix R .
- ❸ $\sigma_j^2(R) \leq \sec \alpha \lambda_j(R(\operatorname{Re} Z)R^*) \leq \sec \alpha \sigma_j(RZR^*)$ for any R and j . Here $\sigma_i(\cdot)$ for singular value, $\lambda_i(\cdot)$ for eigenvalue.
- ❹ $\sigma_j^2(X) \leq \sec \alpha \lambda_j(\operatorname{Re} A) \leq \sec \alpha \sigma_j(A)$ for all $j = 1, \dots, n$.

Construction of a sector through the Cartesian decomposition

Theorem

Let A be an $n \times n$ complex matrix with the Cartesian decomposition $A = M + Ni$, where M is positive definite and N is Hermitian. Then the matrix Z in the sectoral decomposition $A = XZX^$ is determined by the eigenvalues of $M^{-1}N$. Let μ_j be the eigenvalues of $M^{-1}N$ and let $1 + i\mu_j = |1 + i\mu_j|e^{i\gamma_j}$, $|\gamma_j| < \frac{\pi}{2}$, $j = 1, \dots, n$. Then $Z = \text{diag}(e^{i\gamma_1}, \dots, e^{i\gamma_n})$. Set $\gamma(A) = \max_j |\gamma_j|$. Then $W(A) \subseteq S_{\gamma(A)}$.*

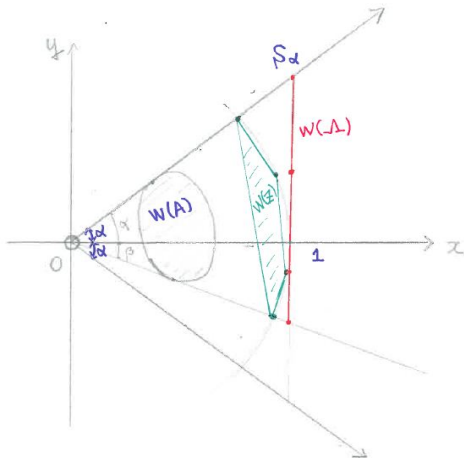
Corollary

Let A be an $n \times n$ complex matrix such that $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then there exist a normal matrix Λ such that $A = (\operatorname{Re} A)^{1/2} \Lambda (\operatorname{Re} A)^{1/2}$. Moreover, $\|\Lambda\|_2 \leq \sec \alpha$ for the spectral norm $\|\cdot\|_2$ on $\mathbb{M}_n$.

Proof. Let $A = M + Ni$ with $M = \operatorname{Re} A$ and $N = \operatorname{Im} A$. Take $\Lambda = I + M^{-1/2} N M^{-1/2} i$. Then Λ is normal. □

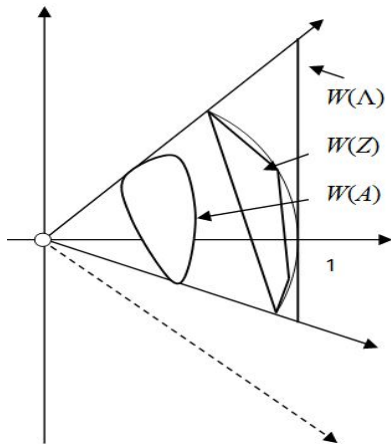
The numerical ranges of A , Z , and Λ

$$A = M + Nj = XZX^* = M^{\frac{k}{2}} \Lambda M^{\frac{k}{2}}$$



$W(A), W(Z), W(\Lambda)$

Rotation



Numerical Ranges of A , Z , Λ

Norm inequalities

Let A be an n -square complex matrix partitioned in the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{where } A_{11} \text{ and } A_{22} \text{ are square.}$$

Lin & Zhou [J. Math. Anal. Appl., 2013, Theorem 3.3]:

Let $A \in \mathbb{M}_n$ be accretive-dissipative ($W(A) \subset$ 1st quadrant) and partitioned as above. Then for any unitarily invariant norm $\|\cdot\|$,

$$\max\{\|A_{12}\|^2, \|A_{21}\|^2\} \leq 4\|A_{11}\| \|A_{22}\|.$$

Q: $4 \rightarrow 2$?

Lemma

Let $A \in \mathbb{M}_n$ have $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then

$$\sigma(A) \prec_w \sec \alpha \lambda(\Re A).$$

Equivalently, for all unitarily invariant norms $\|\cdot\|$ on $\mathbb{M}_n$,

$$\|A\| \leq \sec \alpha \|\Re A\|.$$

Application of the decomposition

Theorem

Let $A \in \mathbb{M}_n$ be partitioned as above and assume $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then for any unitarily invariant norm $\|\cdot\|$,

$$\max\{\|A_{12}\|^2, \|A_{21}\|^2\} \leq \sec^2 \alpha \|A_{11}\| \|A_{22}\|.$$

Proof. Let $A = XZX^*$ and partition X as $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$,

$X_1 \in \mathbb{M}_{p \times n}$. Then $\operatorname{Re} A_{11} = X_1(\operatorname{Re} Z)X_1^*$, $\operatorname{Re} A_{22} = X_2(\operatorname{Re} Z)X_2^*$, and $A_{12} = X_1ZX_2^*$. With (a bit) help of majorization theory,

$$\begin{aligned} \|A_{12}\|^2 &= \|X_1ZX_2^*\|^2 \leq \|X_1X_1^*\| \|X_2Z^*ZX_2^*\| \\ &\leq \sec^2 \alpha \|X_1(\operatorname{Re} Z)X_1^*\| \|X_2(\operatorname{Re} Z)X_2^*\| \\ &= \sec^2 \alpha \|\operatorname{Re} A_{11}\| \|\operatorname{Re} A_{22}\| \\ &\leq \sec^2 \alpha \|A_{11}\| \|A_{22}\|. \quad \square \end{aligned}$$

Corollary

Let $A \in \mathbb{M}_n$ be accretive-dissipative and partitioned as before. Then $W(e^{-\pi/4i} A) \subseteq S_{\pi/4}$ and for any unitarily invariant norm $\|\cdot\|$,

$$\max\{\|A_{12}\|^2, \|A_{21}\|^2\} \leq 2 \|A_{11}\| \|A_{22}\|.$$

Proof. Set $\alpha = \pi/4$ in the theorem. Then $\sec^2 \alpha = 2$. □

Lemma

*Let $H = \begin{bmatrix} H_{11} & * \\ * & H_{22} \end{bmatrix}$ be an $n \times n$ positive semidefinite matrix, where H_{11} and H_{22} are square submatrices (possibly of different sizes). Then*

$$\lambda(H) \prec \lambda(H_{11}) + \lambda(H_{22}).$$

Consequently, for all unitarily invariant norms $\|\cdot\|$ on $\mathbb{M}_n$,

$$\|H\| \leq \|H_{11}\| + \|H_{22}\|.$$

Theorem

Let $A \in \mathbb{M}_n$ be partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and let $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then for any unitarily invariant norm $\|\cdot\|$ on $\mathbb{M}_n$,

$$\|A\| \leq \sec \alpha (\|A_{11}\| + \|A_{22}\|).$$

- the end -