NONSURJECTIVE ZERO PRODUCT PRESERVERS BETWEEN MATRIX SPACES OVER AN ARBITRARY FIELD

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ABSTRACT. A map Φ between matrices is said to be zero product preserving if

$$\Phi(A)\Phi(B) = 0 \quad \text{whenever} \quad AB = 0.$$

In this paper, we give concrete descriptions of an additive/linear zero product preserver $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ between matrix algebras of different dimensions over an arbitrary field $\mathbb{F}$, and $n \geq 2$. In particular, we show that if Φ is linear and preserves zero products then

$$\Phi(A) = S \begin{pmatrix} R_1 \otimes A & 0 \\ 0 & \Phi_0(A) \end{pmatrix} S^{-1},$$

for some invertible matrices R_1 in $\mathbf{M}_k(\mathbb{F})$, S in $\mathbf{M}_r(\mathbb{F})$ and a zero product preserving linear map $\Phi_0 : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_{r-nk}(\mathbb{F})$ into nilpotent matrices. If $\Phi(I_n)$ is invertible, then Φ_0 is vacuous. In general, the structure of Φ_0 could be quite arbitrary, especially when $\Phi_0(\mathbf{M}_n(\mathbb{F}))$ has trivial multiplication, i.e., $\Phi_0(X)\Phi_0(Y) = 0$ for all X, Y in $\mathbf{M}_n(\mathbb{F})$. We show that if $\Phi_0(I_n) = 0$ or $r - nk \leq n + 1$, then $\Phi_0(\mathbf{M}_n(\mathbb{F}))$ indeed has trivial multiplication. More generally, we characterize subspaces $\mathbf{V}$ of square matrices satisfying $XY = 0$ for any $X, Y \in \mathbf{V}$. Similar results for double zero product preserving maps are obtained.

1. INTRODUCTION

Preserver problems of matrices and operators attract a lot of attention; see, for example, [6, 9, 13, 15, 17, 19, 20, 22, 25, 26], and the references therein. Some authors study those additive or linear maps Φ preserving zero products, that is,

$$\Phi(A)\Phi(B) = 0 \quad \text{whenever} \quad AB = 0.$$

See, for example, [1–5, 7, 12, 14]. The classical results of Jacobson, Rickart, Kaplansky, Herstien, etc. (see, e.g., [10, 11]), ensure that every *surjective* zero product preserving linear map $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_n(\mathbb{F})$ between matrices over an arbitrary field $\mathbb{F}$ is a scalar multiple of an inner algebra isomorphism, namely, $A \mapsto \alpha S^{-1}AS$, for a nonzero scalar α and an invertible S in $\mathbf{M}_n(\mathbb{F})$. See, e.g., [6, Theorems 2.6 and 3.1].

The situation is quite different when Φ is not surjective. For example, let f be any map sending $n \times n$ square matrices to $p \times q$ rectangular matrices over any field $\mathbb{F}$. The map $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_{p+q}(\mathbb{F})$ defined by $A \mapsto \begin{pmatrix} 0_p & f(A) \\ 0 & 0_q \end{pmatrix} \in \mathbf{M}_{p+q}(\mathbb{F})$ satisfies that $\Phi(A)\Phi(B) = 0_{p+q}$ for any $A, B \in \mathbf{M}_n(\mathbb{F})$. It trivially preserves zero products. One may impose extra assumptions such as being additive or linear to f so that the map Φ will be additive or linear, respectively. However, Φ is far away from being multiplicative in any case.

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In this paper, we give descriptions of the structures of an additive or linear zero product preserver $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ for arbitrary dimensions n and r. More precisely, suppose $S^{-1}\Phi(I_n)S = R \oplus N$ where $S \in \mathbf{M}_r(\mathbb{F}), R \in \mathbf{M}_s(\mathbb{F})$ are invertible, and $N \in \mathbf{M}_{r-s}(\mathbb{F})$ is nilpotent with nil index ν. If Φ is additive, then $s = nk$ for some integer $k \geq 0$, and Φ assumes the form

$$A \mapsto S \begin{pmatrix} R\Phi_1(A) & 0 \\ 0 & \Phi_0(A) \end{pmatrix} S^{-1},$$

where $\Phi_1 : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_{nk}(\mathbb{F})$ is a ring homomorphism such that $\Phi_1(A)R = R\Phi_1(A)$ for all $A \in \mathbf{M}_n$, and $\Phi_0 : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_{r-nk}(\mathbb{F})$ is an additive zero product preserving map such that $\Phi_0(A)^{\nu+1} = 0$ for every A.

If Φ is linear then, by using a suitable $S \in \mathbf{M}_r(\mathbb{F})$ to write $S^{-1}\Phi(I_n)S = R \oplus N$, we have $R = R_1 \otimes I_n$ for an $R_1 \in \mathbf{M}_k(\mathbb{F})$ such that $R\Phi_1$ assumes the form $A \mapsto (R_1 \otimes I_n)(I_k \otimes A) = R_1 \otimes A$. The map Φ_0 could have wild structure. Nevertheless, we obtain additional information on Φ_0 under some mild assumptions. In particular, if $\Phi(I_n)$ is invertible, Φ assumes the form $A \mapsto S(R_1 \otimes A)S^{-1}$; if $\Phi(I_n)$ is a nilpotent, we have $\Phi = \Phi_0$. More generally, if $\Phi(I_n)$ is diagonalizable, then $N = 0$ and thus $\nu = 1$; in this case, $\Phi_0(A)\Phi_0(B) = 0$ for all $A, B \in \mathbf{M}_n(\mathbb{F})$.

Our paper is organized as follows. In Section 2, we collect some preliminary results. In Section 3, we study additive and linear zero product preservers between matrix algebras of arbitrary sizes over an arbitrary field. We also study linear maps Φ between matrix algebras preserving double zero products, that is,

$$\Phi(A)\Phi(B) = \Phi(B)\Phi(A) = 0 \quad \text{whenever} \quad AB = BA = 0.$$

In this case, Φ assumes the form

$$A \mapsto S \begin{pmatrix} R_1 \otimes A & 0 & 0 \\ 0 & R_2 \otimes A^{\mathrm{t}} & 0 \\ 0 & 0 & \Phi_0(A) \end{pmatrix} S^{-1}$$

for some invertible matrices $R_1 \in \mathbf{M}_p(\mathbb{F}), R_2 \in \mathbf{M}_q(\mathbb{F})$ and $S \in \mathbf{M}_r(\mathbb{F})$. Here the linear map $\Phi_0 : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_{r-n(p+q)}(\mathbb{F})$ preserves double zero products and sends diagonalizable matrices to nilpotent elements. In Section 4, we put attention on additive zero product preserving maps Φ_0 into nilpotent matrices. In particular, we obtain sufficient and necessary conditions to ensure that the range space of such a Φ_0 has trivial multiplication. Finally, we study in details the structure of the range space of Φ_0 when it has trivial multiplication.

2. Preliminaries

In our discussion, we always assume that $\mathbb{F}$ is an arbitrary field, and $\{E_{11}, E_{12}, \ldots, E_{nn}\}$ the standard basis for the algebra $\mathbf{M}_n(\mathbb{F})$ of $n \times n$ matrices over $\mathbb{F}$. In other words, $E_{ij} = e_i {e_j}^{\mathrm{t}}$ where $\{e_1, \ldots, e_n\}$ is the standard basis for the vector space $\mathbb{F}^n$ over $\mathbb{F}$, and A^{t} denotes the transpose of a rectangular matrix A. An idempotent is a matrix E satisfying $E^2 = E$. Two idempotents $E, F \in \mathbf{M}_n(\mathbb{F})$ are disjoint if $EF = FE = 0$. A matrix N is a nilpotent if $N^\nu = 0$ for some positive integer ν.

Lemma 2.1. *Let n be a positive integer with $n \geq 2$.*

(a) *The linear space* $\mathbf{M}_n(\mathbb{F})$ *has the following basis consisting of rank one idempotents*

$$\{E_{jj} : 1 \le j \le n\} \cup \{E_{ii} + E_{ij} : 1 \le i \le n, i \ne j\}.$$

For $i \ne j$, *the matrices* $E_{ii} + E_{ij}$ *and* $E_{jj} - E_{ij}$ *are disjoint rank one idempotents.*

(b) *Every idempotent* A *in* $\mathbf{M}_n(\mathbb{F})$ *is a sum of* k *disjoint rank one idempotents, where* k *is the rank of* A.

(c) *Suppose the characteristic of* $\mathbb{F}$ *is not 2. The sum of two idempotents is an idempotent if and only if they are disjoint.*

(d) *Every non-invertible matrix in* $\mathbf{M}_n(\mathbb{F})$ *is a product of idempotents.*

(e) *The ring* $\mathbf{M}_n(\mathbb{F})$ *is generated by its idempotents.*

Proof. Assertions (a)–(c) can be verified directly. Assertion (d) is shown in [8, Theorem]. Assertion (e) is a consequence of (d) and the fact that every matrix can be written as a sum of rank one matrices. ∎

The following result can be found in [28, Corollary 6.7.2] under the assumption that the characteristic of $\mathbb{F}$ is not 2 and $n \ge 3$. Below, we give a self contained proof of the result without the restrictions.

Theorem 2.2. *Let* $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ *be a ring homomorphism. Then there is an invertible* S *in* $\mathbf{M}_r(\mathbb{F})$ *and a unital ring homomorphism* $\tau : \mathbb{F} \to \mathbf{M}_k(\mathbb{F})$ *with* $r - nk \ge 0$ *such that*

$$\Phi(A) = S[(\sum_{i,j=1}^{n} \tau(a_{ij}) \otimes E_{ij}) \oplus 0_{r-nk}]S^{-1} \quad \text{for all } A = (a_{ij}) \in \mathbf{M}_n(\mathbb{F}).$$

Proof. Replacing Φ with the map $X \mapsto S_1^{-1}\Phi(X)S_1$ for some invertible S_1 in $\mathbf{M}_r(\mathbb{F})$, we can assume that $\Phi(I_n) = I_{r_1} \oplus 0_{r_2}$ with $r = r_1 + r_2$. Since $\Phi(A) = \Phi(I_nAI_n) = \Phi(I_n)\Phi(A)\Phi(I_n)$, we may further assume that $\Phi(I_n) = I_r$. Since $E_{ij}E_{kl} = \delta_{jk}E_{il}$, we have

$$\Phi(E_{ij})\Phi(E_{kl}) = \delta_{jk}\Phi(E_{il}), \quad i, j, k, l = 1, 2, \ldots, n, \tag{2.1}$$

where $\delta_{ij} = 1$ when $i = j$, and 0 otherwise. Moreover,

$$I_r = \Phi(I_n) = \sum_{i=1}^{n} \Phi(E_{ii}).$$

Replacing Φ with the map $X \mapsto S_2^{-1}\Phi(X)S_2$ for some invertible S_2 in $\mathbf{M}_r(\mathbb{F})$, we can assume that the idempotents

$$\Phi(E_{ii}) = 0_{k_1} \oplus \cdots \oplus 0_{k_{i-1}} \oplus I_{k_i} \oplus 0_{k_{i+1}} \oplus \cdots \oplus 0_{k_n}, \quad i = 1, \ldots, n.$$

Here, $k_1 + \cdots + k_n = r$.

Let $m = r - k_1 - k_2$. It follows from (2.1) that

$$\Phi(E_{12}) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \oplus 0_m \quad \text{and} \quad \Phi(E_{21}) = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \oplus 0_m,$$

where B_{ij}, C_{ij} are $k_i \times k_j$ matrices for $i, j = 1, 2$. Since $E_{11}E_{12} = E_{12}$ and $E_{12}E_{11} = 0$, we have B_{11}, B_{22} and B_{21} are all zero matrices. Similarly, C_{11}, C_{22} and C_{12} are also zero matrices.

Hence,

$$\Phi(E_{12}) = \begin{pmatrix} 0 & B_{12} \\ 0 & 0 \end{pmatrix} \oplus 0_m \quad \text{and} \quad \Phi(E_{21}) = \begin{pmatrix} 0 & 0 \\ C_{21} & 0 \end{pmatrix} \oplus 0_m, \tag{2.2}$$

On the other hand, $(E_{12}+E_{21})^2 = E_{11}+E_{22}$ implies

$$\begin{pmatrix} 0 & B_{12} \\ C_{21} & 0 \end{pmatrix}^2 = \begin{pmatrix} B_{12}C_{21} & 0 \\ 0 & C_{21}B_{12} \end{pmatrix} = \begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix}.$$

This ensures $k_1 = k_2$ and $B_{12} = C_{21}^{-1}$. Let $k = k_1$.

Applying a similar argument to other (i,j) pairs, we see that

$$\Phi(E_{jj}) = E_{jj} \otimes I_k, \quad \Phi(E_{ij}) = E_{ij} \otimes B_{ij} \text{ for } i<j, \quad \Phi(E_{ij}) = E_{ij} \otimes B_{ji}^{-1} \text{ for } j<i.$$

In particular, $r/n = k$.

Replacing Φ by the map $X \mapsto S_3\Phi(X)S_3^{-1}$ with $S_3 = I_k \oplus B_{12} \oplus B_{13} \oplus \cdots \oplus B_{1n}$, we can further assume that

$$B_{12} = \cdots = B_{1n} = I_k.$$

Actually, if $n \geq 3$ we also have

$$\Phi(E_{ij}) = E_{ij} \otimes I_k \qquad \text{for all } i,j = 1,\ldots,n.$$

To see this, observe $E_{ij} = (E_{i1} + E_{1j} + E_{ij})^2$ for $1 < i < j$. We thus have

$$\Phi(E_{ij}) = (\Phi(E_{i1}) + \Phi(E_{1j}) + \Phi(E_{ij}))^2.$$

This gives

$$E_{ij} \otimes B_{ij} = (E_{i1} \otimes I_k + E_{1j} \otimes I_k + E_{ij} \otimes B_{ij})^2 = E_{ij} \otimes I_k.$$

Reordering the basic vectors, i.e., applying a permutation similarity, we can assume instead

$$\Phi(E_{ij}) = I_k \otimes E_{ij} \qquad \text{for all } i,j = 1,\ldots,n. \tag{2.3}$$

For any a in $\mathbb{F}$, the matrix $\Phi(aI_n)$ commutes with all $\Phi(E_{ij}) = I_k \otimes E_{ij}$. Thus, $\Phi(aI_n) = \tau(a) \otimes I_n$ for some $\tau(a) \in \mathbf{M}_k(\mathbb{F})$. It is easy to see that $a \mapsto \tau(a)$ is a unital ring homomorphism from $\mathbb{F}$ into $\mathbf{M}_k(\mathbb{F})$. Consequently,

$$\Phi(A) = \sum_{i,j=1}^n \Phi(a_{ij}E_{ij}) = \sum_{i,j=1}^n \Phi(a_{ij}I_n)\Phi(E_{ij}) = \sum_{i,j=1}^n \tau(a_{ij}) \otimes E_{ij}. \qquad \blacksquare$$

When the ring homomorphism Φ in Theorem 2.2 is also linear, we see that

$$\begin{aligned} \Phi(A) &= \sum_{i,j=1}^n \Phi(a_{ij}E_{ij}) = \sum_{i,j=1}^n a_{ij}\Phi(E_{ij}) \\ &= \sum_{i,j=1}^n a_{ij} S\,[I_k \otimes E_{ij}]\,S^{-1} = S[I_k \otimes A]S^{-1}, \quad \forall A = (a_{ij}) \in \mathbf{M}_n(\mathbb{F}). \end{aligned}$$

Consequently, we have

Theorem 2.3. *Suppose $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ is an algebra homomorphism. Then there exist a nonnegative integer k with $t = r - nk \geq 0$, and an invertible matrix S in $\mathbf{M}_r(\mathbb{F})$ such that Φ assumes the form*

$$A \mapsto S \begin{pmatrix} I_k \otimes A & 0 \\ 0 & 0_t \end{pmatrix} S^{-1}. \tag{2.4}$$

One might expect that a ring homomorphism Φ between matrices assumes a form similar to (2.4); namely,

$$A \mapsto \alpha S(A_{\tau_1} \oplus \cdots \oplus A_{\tau_k} \oplus 0)S^{-1} \tag{2.5}$$

for some unital ring endomorphisms $\tau_1, \ldots, \tau_k$ of $\mathbb{F}$. Here, A_τ denotes the matrix $(\tau(a_{ij}))$ when $A = (a_{ij})$. If (2.5) holds and Φ is also linear, then all τ_k are the identity map and (2.5) reduces to (2.4). However, the following example tells that it is not always the case.

Example 2.4. Let $\mathbb{F}$ be a purely transcendental extension over another field $\mathbb{K}$, for example $\mathbb{R}/\mathbb{Q}$. According to [27, Corollary 1' in p. 124], there is a nonzero additive derivation $x \mapsto x'$ of $\mathbb{F}$. Consider the unital ring homomorphism $\tau : \mathbb{F} \to \mathbf{M}_2(\mathbb{F})$ defined by

$$\tau(a) = \begin{pmatrix} a & a' \\ 0 & a \end{pmatrix}. \tag{2.6}$$

Note that $\tau(a)$ is not diagonalizable whenever $a' \neq 0$. Consequently, any ring homomorphism $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ with $r \geq 2n$ defined by τ as in Theorem 2.2 does not assume the form (2.5).

3. Additive and Linear Maps Preserving Zero Products

We study those additive/linear maps $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ preserving zero products, that is,

$$\Phi(A)\Phi(B) = 0_r \qquad \text{whenever } A, B \in \mathbf{M}_n(\mathbb{F}) \text{ satisfy } AB = 0_n.$$

We need the following result, which is known as the *fitting decomposition.*

Lemma 3.1 (See, e.g., [28, Theorem A.0.4])**.** *Every $A \in \mathbf{M}_n(\mathbb{F})$ is similar to a direct sum $R \oplus N$ of an invertible matrix $R \in \mathbf{M}_s(\mathbb{F})$ and a nilpotent matrix $N \in \mathbf{M}_{n-s}(\mathbb{F})$ such that N is a direct sum of upper triangular Jordan blocks for the eigenvalue zero of A. Here, R or N can be vacuous. In particular, if A is diagonalizable (resp. an idempotent), then $N = 0_{n-s}$ (resp. and $R = I_s$).*

By Lemma 3.1, there is an invertible matrix S in $\mathbf{M}_r(\mathbb{F})$ such that

$$S^{-1}\Phi(I_n)S = R \oplus N,$$

where R in $\mathbf{M}_s(\mathbb{F})$ is invertible, and N in $\mathbf{M}_{r-s}(\mathbb{F})$ is nilpotent such that N is a direct sum of upper triangular Jordan blocks for the eigenvalue zero of $\Phi(I_n)$. Furthermore, the size ν of the largest Jordan block of N is the *nil index* of the nilpotent matrix N, which is the smallest positive integer ν such that $N^\nu = 0$. If $S_1^{-1}\Phi(I_n)S_1 = R_1 \oplus N_1$ is another direct sum of an invertible matrix R_1 and a nilpotent matrix N_1 for an invertible matrix S_1, then for any $k \geq 1$,

$$S(R^k \oplus N^k)S^{-1} = (S(R \oplus N)S^{-1})^k = (S_1(R_1 \oplus N_1)S_1^{-1})^k = S_1(R_1^k \oplus N_1^k)S_1^{-1}.$$

Since R, R_1, S, S_1 are all invertible, by counting ranks we conclude that the nilpotent matrices N and N_1 have the same Jordan form. Therefore, we see that N_1, N have the same nil index ν, and R_1, R have the same rank s. It is clear that s is the rank of the $r \times r$ matrix $\Phi(I_n)^r$. When the nil index $\nu = 1$, that is $N = 0$, we say that $\Phi(I_n)$ *has a zero nilpotent part*; in this case, all the zero Jordan blocks of $\Phi(I_n)$ are 1×1.

Theorem 3.2. *Suppose $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ is an additive map preserving zero products, and $n \geq 2$. Then there are invertible matrices S in $\mathbf{M}_r(\mathbb{F})$ and R_1 in $\mathbf{M}_k(\mathbb{F})$ such that Φ has the form*

$$A \mapsto S \begin{pmatrix} (R_1 \otimes I_n)\Phi_1(A) & 0 \\ 0 & \Phi_0(A) \end{pmatrix} S^{-1}. \tag{3.1}$$

Here, $\Phi_1 : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_{nk}(\mathbb{F})$ has the form $(a_{ij}) \mapsto \sum_{ij} \tau(a_{ij}) \otimes E_{ij} \in \mathbf{M}_k \otimes \mathbf{M}_n$ for a unital ring homomorphism $\tau : \mathbb{F} \to \mathbf{M}_k(\mathbb{F})$, such that $R_1\tau(a) = \tau(a)R_1$ for all $a \in \mathbb{F}$, and $\Phi_0 : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_{r-kn}(\mathbb{F})$ is a zero product preserving additive map into nilpotent matrices such that $\Phi_0(I_n)$ commutes with $\Phi_0(A)$ for all $A \in \mathbf{M}_n(\mathbb{F})$, and the product of any $\nu + 1$ elements in $\Phi_0(\mathbf{M}_n(\mathbb{F}))$ is zero if $\Phi_0(I_n)$ has nil index ν.

In particular, $\Phi(I_n) = S[(R_1 \otimes I_n) \oplus \Phi_0(I_n)]S^{-1}$, and the following hold.

(1) *$\Phi(I_n)$ is invertible if and only if Φ_0 is vacuous.*
(2) *$\Phi(I_n) = I_r$ if and only if Φ_0 is vacuous and $R_1 = I_k$.*
(3) *$\Phi(I_n)$ is nilpotent if and only if Φ_1 is vacuous.*
(4) *If $\Phi(I_n)$ has a zero nilpotent part, i.e., $\Phi_0(I_n) = 0$, then $\Phi_0(X)\Phi_0(Y) = 0$ for all $X, Y \in \mathbf{M}_n(\mathbb{F})$.*

We need the following elementary lemma to prove Theorem 3.2. We sketch a proof for easy reference.

Lemma 3.3. *Suppose $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ is an additive map preserving zero products, and $n \geq 2$. Then*

$$\Phi(C)\Phi(AB) = \Phi(CA)\Phi(B) \quad \text{for all} \quad A, B, C \in \mathbf{M}_n(\mathbb{F}).$$

Consequently,

$$\Phi(I_n)\Phi(AB) = \Phi(A)\Phi(B) \quad \text{for all} \quad A, B \in \mathbf{M}_n(\mathbb{F}), \tag{3.2}$$

and

$$\Phi(I_n)\Phi(A) = \Phi(A)\Phi(I_n) \quad \text{for all} \quad A \in \mathbf{M}_n(\mathbb{F}). \tag{3.3}$$

(a) *If $\Phi(I_n)$ is invertible then $A \mapsto \Phi(I_n)^{-1}\Phi(A)$ is a ring homomorphism from $\mathbf{M}_n(\mathbb{F})$ into $\mathbf{M}_r(\mathbb{F})$.*

(b) *If $\Phi(I_n)^\nu = 0$ then the product of any $\nu + 1$ elements from the range of Φ is zero, i.e.,*

$$\Phi(A_1)\Phi(A_2) \cdots \Phi(A_{\nu+1}) = 0 \quad \text{for all} \quad A_1, A_2, \ldots, A_{\nu+1} \in \mathbf{M}_n(\mathbb{F}).$$

In particular, if $\Phi(I_n) = 0$ then the range of Φ has trivial multiplication, i.e.,

$$\Phi(A)\Phi(B) = 0 \quad \text{for all} \quad A, B \in \mathbf{M}_n(\mathbb{F}).$$

Proof. We follow the proof of [6, Lemma 2.1]. Let $E = E^2$ in $\mathbf{M}_n(\mathbb{F})$. For any B, C in $\mathbf{M}_n(\mathbb{F})$, consider

$$(C - CE)EB = CE(B - EB) = 0.$$

By the zero product preserving property, we have

$$(\Phi(C) - \Phi(CE))\Phi(EB) = \Phi(CE)(\Phi(B) - \Phi(EB)) = 0.$$

It follows

$$\Phi(C)\Phi(EB) = \Phi(CE)\Phi(EB) = \Phi(CE)\Phi(B).$$

Hence for any idempotents $E, F, \ldots, G$ in $\mathbf{M}_n(\mathbb{F})$, we have

$$\begin{aligned}\Phi(C)\Phi((EF\cdots G)B) &= \Phi(C)\Phi(E(F\cdots GB)) \\ &= \Phi(CE)\Phi((F\cdots G)B) = \cdots = \Phi(C(EF\cdots G))\Phi(B).\end{aligned}$$

Let $A \in \mathbf{M}_n(\mathbb{F})$ be arbitrary. Since $\mathbf{M}_n(\mathbb{F})$ is generated by its idempotents as a ring by Lemma 2.1, we can write $A = \sum_j E_jF_j\cdots G_j$ as a finite sum of finite products of idempotents. It follows

$$\begin{aligned}\Phi(C)\Phi(AB) &= \sum_j \Phi(C)\Phi((E_jF_j\cdots G_j)B) \\ &= \sum_j \Phi(C(E_jF_j\cdots G_j))\Phi(B) = \Phi(CA)\Phi(B), \quad B, C \in \mathbf{M}_n(\mathbb{F}).\end{aligned}$$

Putting $C = I$, and putting $B = C = I$, respectively, we establish (3.2) and (3.3). It thus follows (a).

For (b), in view of (3.2) and the assumption $\Phi(I_n)^\nu = 0$, we have

$$\begin{aligned}&\Phi(A_1)\Phi(A_2)\Phi(A_3)\cdots\Phi(A_{\nu+1}) = \Phi(I_n)\Phi(A_1A_2)\Phi(A_3)\cdots\Phi(A_{\nu+1}) = \cdots \\ &= \Phi(I_n)^\nu\Phi(A_1A_2A_3\cdots A_{\nu+1}) = 0 \quad \text{for all} \ \ A_1, A_2, \ldots, A_{\nu+1} \in \mathbf{M}_n(\mathbb{F}).\end{aligned}$$

■

Proof of Theorem 3.2. By Lemma 3.1, we may assume that $\Phi(I_n) = S(R\oplus N)S^{-1}$ for invertible matrices $R \in \mathbf{M}_s(\mathbb{F})$, $S \in \mathbf{M}_r(\mathbb{F})$, and a nilpotent matrix $N \in \mathbf{M}_{r-s}(\mathbb{F})$. We may replace Φ by $S^{-1}\Phi(\cdot)S$ and assume that $\Phi(I_n) = R \oplus N$. Let

$$\Phi(X) = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix},$$

where $Y_{11} \in \mathbf{M}_s(\mathbb{F})$. By (3.3) in Lemma 3.3, $\Phi(I_n)\Phi(X) = \Phi(X)\Phi(I_n)$. Therefore,

$$RY_{11} = Y_{11}R, \quad RY_{12} = Y_{12}N, \quad NY_{21} = Y_{21}R \quad \text{and} \quad NY_{22} = Y_{22}N.$$

Without loss of generality, we can assume that $N = \sum_j d_jE_{j,j+1}$ with $d_j \in \{0,1\}$ is a direct sum of upper triangular Jordan blocks of N with zero diagonals. Write $Y_{12} = \big[v_1 \,|\, \cdots \,|\, v_{r-s}\big]$, where $v_1, \ldots, v_{r-s}$ are column vectors. Then

$$\Big[Rv_1 \,|\, Rv_2 \,|\, \cdots \,|\, Rv_{r-s}\Big] = \Big[0 \,|\, d_1v_1 \,|\, \cdots \,|\, d_{r-s-1}v_{r-s-1}\Big].$$

Thus, $v_1 = R^{-1}0 = 0$ and $v_j = d_{j-1}R^{-1}v_{j-1} = 0$ for $j = 2, \ldots, r-s$. Hence, $Y_{12} = 0$. Similarly, we can show that $Y_{21} = 0$. Therefore, $\Phi(X)$ assumes the form $Y_{11} \oplus Y_{22}$. Bringing back the similarity transformation, we can set up the additive maps $\Phi_1 : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_s(\mathbb{F})$ and $\Phi_0 : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_{r-s}(\mathbb{F})$ such that

$$S^{-1}\Phi(X)S = R\Phi_1(X) \oplus \Phi_0(X).$$

Clearly, $\Phi_1(I_n) = R^{-1}R = I_s$. Moreover, $R\Phi_1(A) = \Phi_1(A)R$ for all A in $\mathbf{M}_n(\mathbb{F})$. Suppose $A, B \in \mathbf{M}_n(\mathbb{F})$ such that $AB = 0_n$. Let $S^{-1}\Phi(A)S = A_1 \oplus A_2$ and $S^{-1}\Phi(B)S = B_1 \oplus B_2$. Since $\Phi(A)\Phi(B) = 0_r$, we have $A_1B_1 = 0_s$ and $A_2B_2 = 0_{r-s}$. It also follows

$$R^2\Phi_1(A)\Phi_1(B) = R\Phi_1(A)R\Phi_1(B) = A_1B_1 = 0_s.$$

Consequently, both Φ_1, Φ_0 preserve zero products as well. By Lemma 3.3, Φ_1 is a unital ring homomorphism, and Φ_0 satisfies the said conclusion.

Next, we show that R assumes the form $R_1 \otimes I_n$. Assume that τ is a unital ring homomorphism such that $\Phi_1(A) = \sum_{i,j=1}^n \tau(a_{ij}) \otimes E_{ij}$ for $A = (a_{ij}) \in \mathbf{M}_n(\mathbb{F})$. Let $P \in \mathbf{M}_{nk}(\mathbb{F})$ be the permutation matrix such that $P(X \otimes Y)P^t = Y \otimes X$ for $(X, Y) \in \mathbf{M}_k(\mathbb{F}) \times \mathbf{M}_n(\mathbb{F})$. Then $P\Phi_1(E_{ij})P^t = E_{ij} \otimes \tau(1) = E_{ij} \otimes I_k$, and thus $R\Phi(E_{jj}) = \Phi(E_{jj})R$, for all $i, j = 1, \ldots, n$, implies that $PRP^t = R_{11} \oplus \cdots \oplus R_{nn}$ with $R_{jj} \in \mathbf{M}_k(\mathbb{F})$ for $j = 1, \ldots, n$. Furthermore, the fact $R\Phi(E_{1j}) = \Phi(E_{1j})R$ for $j \neq 1$ implies that $R_{ii} = R_{jj} := R_1$.

One can readily verify assertions (1) – (4). ∎

Theorem 3.4. *Let $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ be a linear map preserving zero products. Then Φ assumes the form*

$$A \mapsto S \begin{pmatrix} R_1 \otimes A & 0 \\ 0 & \Phi_0(A) \end{pmatrix} S^{-1}$$

for some invertible matrices $S \in \mathbf{M}_r(\mathbb{F})$ and $R_1 \in \mathbf{M}_k(\mathbb{F})$ with $r - nk \geq 0$, and a zero product preserving linear map $\Phi_0 : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_{r-nk}(\mathbb{F})$ into nilpotent matrices, as stated in Theorem 3.2. Moreover, if Φ sends rank one idempotents to idempotents then Φ_0 is the zero map.

Proof. The first assertion follows directly from Theorem 3.2. Assume Φ sends rank one idempotents to idempotents. Then $\Phi_0(E) = \Phi_0(E)^\nu = 0$ for every rank one idempotent E in $\mathbf{M}_n(\mathbb{F})$, where $\nu \geq 1$ is the nil index of $\Phi_0(I_n)$. Since every matrix is a linear combination of rank one idempotents by Lemma 2.1, we see that Φ_0 is the zero map. ∎

Example 3.5. The condition that Φ sending rank one idempotents to idempotents in Theorem 3.4 cannot be replaced by the weaker one that $\Phi(I_n)$ being an idempotent. Consider the map $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ defined by

$$\Phi(A) = A \oplus [(a_{11} - a_{22})N], \quad \text{for all } A = (a_{ij}) \in \mathbf{M}_n(\mathbb{F}),$$

with any field $\mathbb{F}$, any $r - 1 > n > 1$, and any $N \in \mathbf{M}_{r-n}(\mathbb{F})$ with $N^2 = 0$. The injective linear map Φ preserves zero products, and $\Phi(I_n) = I_n \oplus 0_{r-n}$ is an idempotent. Although the range of $\Phi_0(A) = (a_{11} - a_{22})N$ has trivial multiplication, $\Phi_0 \neq 0$.

We can use the above results and techniques to study linear double zero product preservers, i.e., those maps Φ between matrices such that

$$\Phi(A)\Phi(B) = \Phi(B)\Phi(A) = 0 \quad \text{whenever} AB = BA = 0.$$

To this end, we also need the following result in [16].

Theorem 3.6. *Let $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ be a linear map. Assume that Φ sends disjoint rank one idempotents to disjoint idempotents.*

(a) *Suppose* $\mathbf{M}_n(\mathbb{F}) \neq \mathbf{M}_2(\mathbb{Z}_2)$. *Then there are nonnegative integers* p, q *with* $s = n(p+q) \leq r$, *and an invertible matrix* S *in* $\mathbf{M}_r(\mathbb{F})$ *such that* Φ *assumes the form*

$$A \mapsto S \begin{pmatrix} I_p \otimes A & & \\ & I_q \otimes A^{\mathrm{t}} & \\ & & 0_{r-s} \end{pmatrix} S^{-1} \quad \text{for all } A \in \mathbf{M}_n(\mathbb{F}). \tag{3.4}$$

(b) *Suppose* $\mathbf{M}_n(\mathbb{F}) = \mathbf{M}_2(\mathbb{Z}_2)$. *Then there are nonnegative integers* k_1, k_2 *with* $k_1 + k_2 \leq r$ *and an invertible matrix* S *in* $\mathbf{M}_r(\mathbb{Z}_2)$ *such that* Φ *assumes the form*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto S \left[\begin{pmatrix} aI_{k_1} + bB_{11} + cC_{11} & bB_{12} + cC_{12} \\ bB_{21} + cC_{21} & dI_{k_2} + bB_{22} + cC_{22} \end{pmatrix} \oplus 0_{r-k_1-k_2} \right] S^{-1},$$

where B_{ij}, C_{ij} *are rectangular* $k_i \times k_j$ *matrices for* $i, j = 1, 2$ *satisfying that*

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}^2 = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}^2 = \begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix}.$$

Conversely, if Φ *assumes the stated form in either case, then* Φ *sends disjoint rank one idempotents to disjoint idempotents.*

Theorem 3.7. *Let* $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ *be a linear map preserving double zero products, i.e.,*

$$\Phi(A)\Phi(B) = \Phi(B)\Phi(A) = 0 \quad \text{whenever} \quad A, B \in \mathbf{M}_n(\mathbb{F}) \text{ satisfies } AB = BA = 0.$$

Then there exist nonnegative integers p, q *such that* $\Phi(I_n)^r$ *has rank* $s = n(p+q)$, *and invertible matrices* R_1 *in* $\mathbf{M}_p(\mathbb{F})$ *and* R_2 *in* $\mathbf{M}_q(\mathbb{F})$ *such that* Φ *assumes the form*

$$A \mapsto S \begin{pmatrix} R_1 \otimes A & 0 & 0 \\ 0 & R_2 \otimes A^{\mathrm{t}} & 0 \\ 0 & 0 & \Phi_0(A) \end{pmatrix} S^{-1}.$$

Here, $\Phi_0 : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_{r-s}(\mathbb{F})$ *is a double zero product preserving linear map satisfying that* $\Phi_0(I_n)$ *commutes with* $\Phi_0(A)$ *for all* $A \in \mathbf{M}_n(\mathbb{F})$ *and* $\Phi_0(A)$ *is a nilpotent matrix for every diagonalizable* A *in* $\mathbf{M}_n(\mathbb{F})$.

When $\Phi(I_n)$ *is invertible,* Φ *assumes the form*

$$A \mapsto S[(R_1 \otimes A) \oplus (R_2 \otimes A^{\mathrm{t}})]S^{-1};$$

when Φ *sends rank one idempotents to idempotents,* Φ *assumes the form*

$$A \mapsto S[(R_1 \otimes A) \oplus (R_2 \otimes A^{\mathrm{t}}) \oplus 0]S^{-1}.$$

Proof. Observe that for any idempotent E in $\mathbf{M}_n(\mathbb{F})$, we have

$$E(I_n - E) = (I_n - E)E = 0.$$

Thus

$$\Phi(E)(\Phi(I_n) - \Phi(E)) = (\Phi(I_n) - \Phi(E))\Phi(E) = 0.$$

This gives

$$\Phi(E)\Phi(I_n) = \Phi(E)^2 = \Phi(I_n)\Phi(E). \tag{3.5}$$

Since every A in $\mathbf{M}_n(\mathbb{F})$ is a linear combination of idempotents by Lemma 2.1,

$$\Phi(I_n)\Phi(A) = \Phi(A)\Phi(I_n) \quad \text{for all } A \in \mathbf{M}_n(\mathbb{F}). \tag{3.6}$$

As argued in the proof of Theorem 3.2, we write

$$S^{-1}\Phi(\cdot)S = R\Phi_1 \oplus \Phi_0$$

in which $R \in \mathbf{M}_s(\mathbb{F})$ is invertible and Φ_1 is a unital linear map from $\mathbf{M}_n(\mathbb{F})$ into $\mathbf{M}_s(\mathbb{F})$ such that $R\Phi_1(A) = \Phi_1(A)R$ for all $A \in \mathbf{M}_n(\mathbb{F})$, while $\Phi_0 : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_{r-s}(\mathbb{F})$ is linear. Both Φ_1, Φ_0 preserve double zero products. In particular, it follows from (3.6) that $\Phi_1(I_n)$ and $\Phi_0(I_n)$ commute with $\Phi_1(A)$ and $\Phi_0(A)$ for all $A \in \mathbf{M}(\mathbb{F})$, respectively.

Since $\Phi_0(I_n)^\nu = 0$ for some positive integer ν, it follows from (3.5) that $\Phi_0(E)^{\nu+1} = \Phi_0(I_n)^\nu\Phi_0(E) = 0$ for all idempotents E in $\mathbf{M}_n(\mathbb{F})$. Let $A \in \mathbf{M}_n(\mathbb{F})$ be diagonalizable, namely, $A = \sum_j a_jE_j$ is a linear sum of disjoint idempotents. It follows from the double zero product preserving property of the linear map Φ_0 that

$$\Phi_0(A)^{\nu+1} = (\sum_j a_j\Phi_0(E_j))^{\nu+1} = \sum_j a_j^{\nu+1}\Phi_0(E_j)^{\nu+1} = 0.$$

On the other hand, we see from (3.5) that Φ_1 preserves idempotents, and indeed, Φ_1 sends disjoint idempotents to disjoint idempotents. Suppose that $\mathbf{M}_n(\mathbb{F}) = \mathbf{M}_2(\mathbb{Z}_2)$. By Theorem 3.6(b), Φ_1 assumes the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} aI_{k_1} + bB_{11} + cC_{11} & bB_{12} + cC_{12} \\ bB_{21} + cC_{21} & dI_{k_2} + bB_{22} + cC_{22} \end{pmatrix}.$$

Since $E_{12}^2 = E_{21}^2 = 0$ and Φ_1 preserves square zero matrices,

$$\Phi_1(E_{12}) = \begin{pmatrix} 0_{k_1} & B_{12} \\ B_{21} & 0_{k_2} \end{pmatrix}, \quad \Phi_1(E_{21}) = \begin{pmatrix} 0_{k_1} & C_{12} \\ C_{21} & 0_{k_2} \end{pmatrix},$$
$$B_{12}B_{21} = C_{12}C_{21} = 0_{k_1} \quad \text{and} \quad B_{21}B_{12} = C_{21}C_{12} = 0_{k_2}.$$

Since $(E_{11} + E_{12} + E_{21} + E_{22})^2 = 0$, we have

$$\begin{pmatrix} I_{k_1} & B_{12} + C_{12} \\ B_{21} + C_{21} & I_{k_2} \end{pmatrix}^2 = 0,$$

and thus

$$I_{k_1} = (B_{12} + C_{12})(B_{21} + C_{21}),$$
$$I_{k_2} = (B_{21} + C_{21})(B_{12} + C_{12}).$$

It follows $k_1 = k_2 := k$ and $(B_{12} + C_{12}) = (B_{21} + C_{21})^{-1}$.

Define $\Psi : \mathbf{M}_2(\mathbb{Z}_2) \to \mathbf{M}_s(\mathbb{Z}_2)$ by

$$\Psi(\cdot) = \big(I_k \oplus (B_{12} + C_{12})\big)\Phi_1(\cdot)\big(I_k \oplus (B_{12} + C_{12})^{-1}\big).$$

Then

$$\Psi(E_{11}) = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Psi(E_{22}) = \begin{pmatrix} 0 & 0 \\ 0 & I_k \end{pmatrix},$$
$$\Psi(E_{12}) = \begin{pmatrix} 0 & B'_{12} \\ B'_{21} & 0 \end{pmatrix}, \quad \Psi(E_{21}) = \begin{pmatrix} 0 & C'_{12} \\ C'_{21} & 0 \end{pmatrix}$$

for some $B'_{12}, B'_{21}, C'_{12}, C'_{21} \in \mathbf{M}_k(\mathbb{Z}_2)$ such that

$$\Psi(E_{12}+E_{21}) = \begin{pmatrix} 0 & B'_{12}+C'_{12} \\ B'_{21}+C'_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix}, \quad \text{and}$$
$$B'_{12}B'_{21} = C'_{12}C'_{21} = B'_{21}B'_{12} = C'_{21}C'_{12} = 0_k.$$

By Lemma 3.1, there is an invertible $U \in \mathbf{M}_k(\mathbb{Z}_2)$ such that

$$UB'_{12}U^{-1} = \begin{pmatrix} R_{12} & 0 \\ 0 & N_{12} \end{pmatrix}$$

for an invertible $R_{12} \in \mathbf{M}_p(\mathbb{Z}_2)$ and a nilpotent $N_{12} \in \mathbf{M}_q(\mathbb{Z}_2)$ with $p+q=k$. Since $B'_{12}B'_{21} = B'_{21}B'_{12} = 0_k$, we see that $UB'_{21}U^{-1} = 0_p \oplus T$ with $T \in \mathbf{M}_q(\mathbb{Z}_2)$ satisfying $TN_{12} = N_{12}T = 0_q$. Since $B'_{12}+C'_{12} = B'_{21}+C'_{21} = I_k$, we have

$$UC'_{12}U^{-1} = (I_p - R_{12}) \oplus (I_q - N_{12}) \quad \text{and} \quad UC'_{21}U^{-1} = I_p \oplus (I_q - T).$$

Since $(I_q - N_{12}) \in \mathbf{M}_q(\mathbb{Z}_2)$ is invertible, and

$$0_k = C'_{12}C'_{21} = (UC'_{12}U^{-1})(UC'_{21}U^{-1}) = (I_p - R_{12})I_p \oplus (I_q - N_{12})(I_q - T),$$

we see that $R_{12} = I_p$ and $T = I_q$, and thus $N_{12} = 0_q$.

Let $\Psi_1(\cdot) = (U \oplus U)\Psi(\cdot)(U \oplus U)^{-1}$. Then,

$$\Psi_1(E_{11}) = \begin{pmatrix} I_p & 0 & 0_p & 0 \\ 0 & I_q & 0 & 0_q \\ 0_p & 0 & 0_p & 0 \\ 0 & 0_q & 0 & 0_q \end{pmatrix}, \quad \Psi_1(E_{22}) = \begin{pmatrix} 0_p & 0 & 0_p & 0 \\ 0 & 0_q & 0 & 0_q \\ 0_p & 0_q & I_p & 0 \\ 0 & 0_q & 0 & I_q \end{pmatrix},$$
$$\Psi_1(E_{12}) = \begin{pmatrix} 0_p & 0 & I_p & 0 \\ 0 & 0_q & 0 & 0_q \\ 0_p & 0 & 0_p & 0 \\ 0 & I_q & 0 & 0_q \end{pmatrix}, \quad \Psi_1(E_{21}) = \begin{pmatrix} 0_p & 0 & 0_p & 0 \\ 0 & 0_q & 0 & I_q \\ I_p & 0 & 0_p & 0 \\ 0 & 0_q & 0 & 0_q \end{pmatrix}.$$

Consequently, Ψ_1 assumes the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} aI_p & 0 & bI_p & 0 \\ 0 & aI_q & 0 & cI_q \\ cI_p & 0 & dI_p & 0 \\ 0 & bI_q & 0 & dI_q \end{pmatrix}.$$

After a permutation similarity,

$$V\Psi_1(A)V^{-1} = \begin{pmatrix} I_p \otimes A & 0 \\ 0 & I_q \otimes A^{\mathrm{t}} \end{pmatrix}.$$

Let $W = \big(I_k \oplus (B_{12}+C_{12})\big)^{-1}(U \oplus U)^{-1}V^{-1} \in \mathbf{M}_{n(p+q)}(\mathbb{Z}_2)$. Then Φ_1 assumes the form

$$\Phi_1(A) = W \begin{pmatrix} I_p \otimes A & 0 \\ 0 & I_q \otimes A^{\mathrm{t}} \end{pmatrix} W^{-1}.$$

Since $R\Phi_1(A) = \Phi_1(A)R$, we have

$$(W^{-1}RW)\begin{pmatrix} I_p \otimes A & 0 \\ 0 & I_q \otimes A^{\mathrm{t}} \end{pmatrix} = \begin{pmatrix} I_p \otimes A & 0 \\ 0 & I_q \otimes A^{\mathrm{t}} \end{pmatrix}(W^{-1}RW) \quad \text{for all } A \in \mathbf{M}_2(\mathbb{Z}_2).$$

Hence,

$$W^{-1}RW = \begin{pmatrix} R_1 \otimes I_n & 0 \\ 0 & R_2 \otimes I_n \end{pmatrix}$$

for some invertible matrices $R_1 \in \mathbf{M}_p(\mathbb{Z}_2)$ and $R_2 \in \mathbf{M}_q(\mathbb{Z}_2)$. Replacing S by $S(W \oplus I_{r-n(p+q)})$, we see that Φ assumes the asserted form.

If $\mathbf{M}_n(\mathbb{F}) \neq \mathbf{M}_2(\mathbb{Z}_2)$ then, by Theorem 3.6(a), we can assume that Φ_1 also carries the form $\Phi_1(A) = W[(I_p \otimes A) \oplus (I_q \otimes A^{\mathrm{t}})]W^{-1}$ for some invertible matrix $W \in \mathbf{M}_{n(p+q)}(\mathbb{F})$. A similar argument derives that $W^{-1}RW = (R_1 \otimes I_n) \oplus (R_2 \otimes I_n)$, and thus Φ assumes the stated form by replacing S with $S(W \oplus I_{r-n(p+q)})$.

Finally, if $\Phi(I_n) = R$ is invertible, then $\Phi(A) = S[(R_1 \otimes A) \oplus (R_2 \otimes A^{\mathrm{t}})]S^{-1}$ for all $A \in \mathbf{M}_n(\mathbb{F})$, as asserted. If Φ sends rank one idempotents to idempotents, then Φ_0 sends any rank one idempotent to a nilpotent idempotent, and thus a zero matrix. Since every matrix is a linear combination of rank one idempotent by Lemma 2.1, Φ_0 is a zero map. ■

Remark 3.8. In Theorem 3.7, more can be said about Φ_0 if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, namely, one can conclude that

$$\Phi_0(X)^{\nu+1} = 0 \quad \text{for all } X \in \mathbf{M}_n(\mathbb{F}).$$

To see this, first note that we have $\Phi_0(A)^{\nu+1} = 0$ for every diagonalizable A. If $\mathbb{F} = \mathbb{C}$ then the assertion follows by noting that diagonalizable matrices are dense in $\mathbf{M}_n(\mathbb{C})$ and linear maps are continuous.

When $\mathbb{F} = \mathbb{R}$, we need a more delicate argument which works for both the real and complex cases. For a given $A = (a_{ij}) \in \mathbf{M}_n(\mathbb{F})$, we can write $A = B + C$ with $B = \operatorname{diag}(L, 2L, \ldots, nL)$ for sufficiently large $L > 0$ so that the n Gershgorin disks of $C = A - B$

$$D_j = \{z : |z - a_{jj} + jL| \le \sum_{k \neq j} |a_{jk}|\}, \quad j = 1, \ldots, n$$

are disjoint. For instance, this will hold if $L > 2\sum_{k=1}^n |a_{jk}|$ for all $j = 1, \ldots, n$. In such a case the centers of any two disks D_j and D_ℓ will satisfy

$$|(a_{jj} - jL) - (a_{\ell\ell} - \ell L)| \ge |(j - \ell)L| - |a_{jj}| - |a_{\ell\ell}| > \sum_{k \neq j} |a_{jk}| + \sum_{k \neq \ell} |a_{\ell k}|$$

which are the sum of the radii of the two disks. So the two disks are disjoint. Therefore, there is one eigenvalue in every disk D_j for $j = 1, \ldots, n$. If $\mathbb{F} = \mathbb{R}$, then the complex eigenvalues of C will occur in conjugate pairs. Since the disjoint circular disks $D_1, \ldots, D_n$ have centers $a_{11} - L, a_{22} - 2L, \ldots, a_{nn} - nL$ on the real line, we see that C has n distinct real eigenvalues. Thus, in both the real and complex cases, B, C are diagonalizable. Consequently, $\Phi(B)^{\nu+1} = \Phi(C)^{\nu+1} = 0$. Moreover, for $t = 2, \ldots, \nu + 1$, one can use the above argument to show that $tB + C$ has n disjoint Gershgorin disks, and hence has n distinct eigenvalues in $\mathbb{F}$. Thus,

$$0 = \Phi(tB + C)^{\nu+1} = (t\Phi(B) + \Phi(C))^{\nu+1} = \sum_{k=0}^{\nu+1} t^k G_k(\Phi(B), \Phi(C)),$$

where $G_k(\Phi(B), \Phi(C))$ is the sum of $\binom{\nu+1}{k}$ matrices, and each summand is a product of $\nu + 1$ matrices with k of them equal to $\Phi(B)$ and the rest equal to $\Phi(C)$. In particular,

$G_{\nu+1}(\Phi(B),\Phi(C)) = \Phi(B)^{\nu+1} = 0$ and $G_0(\Phi(B),\Phi(C)) = \Phi(C)^{\nu+1} = 0$. So,

$$0 = \sum_{k=1}^{\nu} t^k G_k(\Phi(B),\Phi(C)), \quad t = 0, 2, \ldots, \nu+1.$$

Now, the matrix-coefficient polynomial in t has degree at most ν, and has $\nu+1$ zeros. So, $G_k(\Phi(B),\Phi(C)) = 0$ for all $k = 1,\ldots,\nu$, and thus $\Phi(A)^{\nu+1} = \Phi(B+C)^{\nu+1} = 0$.

By the above discussion, we see that in Theorem 3.7, even if no extra information on $\mathbb{F}$ is given, one can conclude that $\Phi_0(A)^{\nu+1} = 0$ whenever $A = B+C$ such that $B, C, t_1B+C, \ldots, t_\nu B+C$ are diagonalizable for ν distinct nonzero elements $t_1,\ldots,t_\nu \in \mathbb{F}$.

In the proof of Theorem 3.7, one can deduce from (3.5) that the unital double zero product linear preserver Φ_1 sends idempotents to idempotents. When the underlying field $\mathbb{F}$ does *not* have characteristic 2, a routine argument will show that Φ_1 is a Jordan homomorphism, as well as a direct sum of a homomorphism and an anti-homomorphism, and Theorem 2.3 applies. However, the following example tells us that a linear Jordan homomorphism between matrices over a field of characteristic 2 does not necessarily assume the form (3.4). Therefore, Theorem 3.6, together with a separate treatment to the case when $\mathbf{M}_n(\mathbb{F}) = \mathbf{M}_2(\mathbb{Z}_2)$, is necessary in proving Theorem 3.7.

Example 3.9. Let $\mathbb{F}$ be a field of characteristic 2, $n \geq 2$ and $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ defined by

$$A \mapsto \operatorname{trace}(A) I_r.$$

For any $A, B \in \mathbf{M}_n(\mathbb{F})$, observe that both

$$\operatorname{trace}(AB+BA) = 2\operatorname{trace}(AB) = 0, \quad \text{and}$$

$$\operatorname{trace}(A)\operatorname{trace}(B) + \operatorname{trace}(B)\operatorname{trace}(A) = 2\operatorname{trace}(A)\operatorname{trace}(B) = 0.$$

Hence, Φ is a linear Jordan homomorphism. Since the trace of an idempotent in $\mathbf{M}_n(\mathbb{F})$ is either zero or one (modulo 2), Φ also sends idempotents to idempotents. But Φ does not assume the form (3.4).

Note that Φ does not send disjoint idempotents to disjoint idempotents, and thus Theorem 3.6 does not apply.

4. Zero product preserving maps into nilpotents

By Theorem 3.2, every zero product preserving additive map $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ assumes the form

$$A \mapsto S(R\Phi_1(A) \oplus \Phi_0(A))S^{-1} = S(\Phi_1(A)R \oplus \Phi_0(A))S^{-1}, \tag{4.1}$$

where R, S are invertible matrices, $\Phi_1 : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_{nk}(\mathbb{F})$ is a unital ring homomorphism and $\Phi_0 : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_{r-nk}(\mathbb{F})$ is a zero product preserving additive map sending matrices to nilpotent matrices. With the discussion in Section 3, we have a good understanding of Φ_1. In this section, we focus on Φ_0. By Theorem 3.2, if $\Phi_0(I_n) = 0$, then $\Phi_0(\mathbf{M}_n(\mathbb{F}))$ has trivial multiplication.

In [3, Theorem 5.2], it is shown that every zero product preserving additive map Φ : $\mathbf{M}_n(\mathbf{D}) \to \mathbf{M}_n(\mathbf{D})$ of matrices over a division ring $\mathbf{D}$ either has a range with trivial multiplication, or $\Phi(\cdot) = C\Psi(\cdot) = \Psi(\cdot)C$ for a ring endomorphism Ψ and a matrix C. However, such conclusion may not hold for maps between matrices of different sizes. For instance, we can have the following example based on [21, p. 310] and [6, Example 2.5].

Example 4.1. Consider $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ with $r \geq n+2 \geq 4$ defined by

$$(a_{ij}) \mapsto \begin{pmatrix} 0 & a_{11} & \cdots & a_{1n} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & a_{1n} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & a_{nn} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The linear map Φ preserves zero products. Note that $\Phi(I_n)^2 = 0$, and thus any product of three elements in $\Phi(\mathbf{M}_n(\mathbb{F}))$ is zero. Since $\Phi(E)^2 \neq 0$ with $E = E_{11} + E_{1n}$, the image of Φ has a nontrivial multiplication.

We claim that Φ cannot be written in the form $C\Psi$ for any C in $\mathbf{M}_r(\mathbb{F})$ and any homomorphism $\Psi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$. Assume on the contrary that $\Phi = C\Psi$. Then we get a contradiction since

$$\begin{aligned} \Phi(E)^2 &= \Phi(E)C\Psi(E) = \Phi(E)C\Psi(E_{11}E) = \Phi(E)C\Psi(E_{11})\Psi(E) \\ &= \Phi(E)\Phi(E_{11})\Psi(E) = 0\Psi(E) = 0. \end{aligned}$$

We have the following.

Proposition 4.2. *Suppose that $1 \leq r \leq n+1$ and $n \geq 2$. Let $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ be an additive zero product preserver such that $\Phi(I_n)$ is a nilpotent matrix. Then $\Phi(X)\Phi(Y) = 0$ for any $X, Y \in \mathbf{M}_n(\mathbb{F})$.*

By the above proposition and Theorem 3.2, we have the following counterpart of [3, Theorem 5.2].

Corollary 4.3. *Suppose that $1 \leq r \leq n+1$ and $n \geq 2$. Let $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ be an additive zero product preserver.*

(a) *Assume that $\Phi(I_n)$ is not a nilpotent. Then $r = n$ or $r = n+1$, and Φ assumes the form*
$$A \mapsto \alpha S(A_\tau \oplus 0_{r-n})S^{-1} \tag{4.2}$$
for some nonzero scalar α, an invertible matrix S in $\mathbf{M}_r(\mathbb{F})$, and a unital ring endomorphism τ of $\mathbb{F}$. Here, $A_\tau = (\tau(a_{ij}))$ if $A = (a_{ij})$.

(b) *Assume that $\Phi(I_n)$ is a nilpotent. Then the range of Φ always has trivial multiplication if either*
   i. *$\mathbb{F}$ is not an infinite field of characteristic 2, or*
   ii. *$\mathbb{F}$ is an infinite field of characteristic 2 and Φ is $\mathbb{F}$-linear.*

To prove Proposition 4.2, we need the following lemma. It provides us a sufficient and necessary condition for $\Phi_0(\mathbf{M}_n(\mathbb{F}))$ having trivial multiplication.

Lemma 4.4. *Let $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ be an additive zero product preserver, and $n \geq 2$. When $\mathbb{F}$ is an infinite field of characteristic 2, we assume in addition that Φ is $\mathbb{F}$-linear. The range of Φ has trivial multiplication exactly when Φ sends every scalar multiple of a rank one idempotent to a square zero element.*

Proof. We verify the sufficiency only. By Lemma 2.1, for every X, Y in $\mathbf{M}_n(\mathbb{F})$ we can write their product as a linear combination of idempotents, $XY = \sum_j \beta_j E_j$, say. In the case when 2 is invertible in $\mathbb{F}$, we see that each scalar $\beta = \left(\frac{\beta+1}{2}\right)^2 - \left(\frac{\beta-1}{2}\right)^2$. In the case when $\mathbb{F}$ is a finite field of characteristic 2, the map $\beta \mapsto \beta^2$ is injective, and thus bijective, from $\mathbb{F}$ onto $\mathbb{F}$. Thus in both cases we can assume that $\beta_k = \alpha_k^2 - \gamma_k^2$ for some α_k, γ_k in $\mathbb{F}$ for all k.

If Φ is assumed additive and $\mathbb{F}$ is not an infinite field of characteristic 2, then with (3.2) we have

$$\begin{aligned}\Phi(X)\Phi(Y) = \Phi(I_n)\Phi(XY) &= \sum_j \Phi(I_n)\Phi((\alpha_j^2 - \gamma_j^2)E_j) \\ &= \sum_j \Phi(I_n)\Phi((\alpha_j E_j)^2) - \sum_j \Phi(I_n)\Phi((\gamma_j E_j)^2) \\ &= \sum_j \Phi(\alpha_j E_j)^2 - \sum_j \Phi(\gamma_j E_j)^2 = 0.\end{aligned}$$

For the exceptional case that $\mathbb{F}$ is an infinite field of characteristic 2, with the linearity of Φ it follows from (3.2) that

$$\begin{aligned}\Phi(X)\Phi(Y) = \Phi(I_n)\Phi(XY) &= \Phi(I_n)\Phi(\sum_j \beta_j E_j) \\ &= \sum_j \beta_j \Phi(I_n)\Phi(E_j) = \sum_j \beta_j \Phi(E_j)^2 = 0.\end{aligned}$$

■

Proof of Proposition 4.2. Let $\Phi(I_n)$ be a nilpotent. Suppose on the contrary that $\Phi(\mathbf{M}_n(\mathbb{F}))$ does not have trivial multiplication. By Lemma 4.4, $\Phi(\alpha E)^2 \neq 0$ for a rank one idempotent E in $\mathbf{M}_n(\mathbb{F})$ and $\alpha \neq 0$ in $\mathbb{F}$. Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{F}^n$ consisting of eigenvectors of E such that $Ee_1 = e_1$ and $Ee_j = 0$ for $j = 2, \ldots, n$. In this setting, we can assume $E = E_{11}$.

Because Φ preserves zero products,

$$\Phi(\alpha E_{ij})\Phi(\alpha E_{kl}) = 0, \quad \text{whenever } j \neq k, \text{ and } i, j, k, l = 1, \ldots, n. \tag{4.3}$$

Observe also that

$$(\alpha E_{11} + \alpha E_{1j})(\alpha E_{11} - \alpha E_{j1}) = 0$$

implies

$$\Phi(\alpha E_{1j})\Phi(\alpha E_{j1}) = \Phi(\alpha E_{11})^2 \neq 0, \qquad j = 1, \ldots, n. \tag{4.4}$$

Since $\Phi(I_n)$ is a nilpotent, $\Phi(\alpha E_{11})$ is a nilpotent as well by Lemma 3.3(b). After a similarity transformation, we can assume that $\Phi(\alpha E_{11}) = J_1 \oplus \cdots \oplus J_m$ is a direct sum of its Jordan blocks, all of which have zero diagonals. Since $\Phi(\alpha E_{11})^2 \neq 0$, we can further assume that J_1 is of size at least 3; namely,

$$J_1 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since $E_{1j}E_{11} = E_{11}E_{j1} = 0$, we see that the first and the second columns of $\Phi(\alpha E_{1j})$ are zero columns, and the second and the third rows of $\Phi(\alpha E_{j1})$ are zero rows for $j = 2, \ldots, n$.

Denote by R_j the first row of $\Phi(\alpha E_{1j})$, and by C_j the third column of $\Phi(\alpha E_{j1})$ for $j = 2, 3, \ldots, n$. Let

$$R = \begin{pmatrix} R_2 \\ R_3 \\ \vdots \\ R_n \end{pmatrix}_{(n-1)\times r} \quad \text{and} \quad C = \begin{pmatrix} C_2 & C_3 & \cdots & C_n \end{pmatrix}_{r\times(n-1)}.$$

The conditions (4.3) and (4.4) tell us that $R_iC_j = 1$ whenever $i = j$, and 0 whenever $i \neq j$. In other words, $RC = I_{n-1}$. Note that the first and second columns of R are both the zero columns. On the other hand, since the third row of C is the zero row, we can replace the third column of R by the zero column to get a new $(n-1)\times r$ matrix R' such that $R'C = RC = I_{n-1}$. Therefore, R' has rank at least $n-1$. Since the first three columns of R' are zero, we have $r - 3 \geq n - 1$. This contradiction establishes the assertion. ∎

The unital ring homomorphism from $\mathbb{F}$ into $\mathbf{M}_2(\mathbb{F})$ given by (2.6) in Example 2.4 shows that Proposition 4.2 does not hold when $r = 2n = 2$.

The following theorem demonstrates the structure of the range space $\mathbf{V} = \Phi(\mathbf{M}_n(\mathbb{F}))$ when $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ is a linear map such that $\Phi(A)\Phi(B) = 0$ for all $A, B \in \mathbf{M}_n(\mathbb{F})$. In particular, $\Phi(\mathbf{M}_n(\mathbb{F}))$ has dimension at most $r^2/4$ in this case.

Theorem 4.5. *Let* $\mathbf{V}$ *be a vector subspace of* $\mathbf{M}_r(\mathbb{F})$.

(a) *Suppose that there is an invertible matrix* S *in* $\mathbf{M}_r(\mathbb{F})$ *such that* $S^{-1}\mathbf{V}S$ *consists of matrices of the form*

$$\begin{pmatrix} 0_p & Z_{12} & XB_1 \\ 0_p & 0_p & 0 \\ 0 & B_2Y & 0_q \end{pmatrix} \quad \text{with } Z_{12} \in \mathbf{M}_p(\mathbb{F}),\ X \in \mathbf{M}_{p,k_1}(\mathbb{F}) \text{ and } Y \in \mathbf{M}_{k_2,p}(\mathbb{F}), \tag{4.5}$$

for some nonnegative integers p, k_1, k_2 *with* $k_1 + k_2 \leq q = r - 2p$, *and fixed matrices* $B_1 \in \mathbf{M}_{k_1,q}(\mathbb{F})$ *and* $B_2 \in \mathbf{M}_{q,k_2}(\mathbb{F})$ *with* $B_1B_2 = 0_{k_1,k_2}$. *Then* $\mathbf{V}$ *has trivial multiplication. Moreover, the dimension of* $\mathbf{V}$ *is at most* $p(r-p) \leq r^2/4$.

(b) *If* $\mathbb{F}$ *has more than* $r/2$ *elements and* $\mathbf{V}$ *has trivial multiplication, then we can represent* $\mathbf{V}$ *as in* (4.5) *such that* p *is the maximal rank of matrices in* $\mathbf{V}$, *and* B_1, B_2 *are full rank matrices.*

Proof. (a) Since $B_1B_2 = 0$, the product $ZZ' = 0$ for all $Z, Z' \in \mathbf{V}$. Moreover, since a matrix in $\mathbf{V}$ is determined by $Z_{12} \in \mathbf{M}_p(\mathbb{F})$, $X \in \mathbf{M}_{p,k_1}(\mathbb{F})$ and $Y \in \mathbf{M}_{k_2,p}(\mathbb{F})$, the dimension of $\mathbf{V}$ is bounded by

$$p^2 + p(k_1 + k_2) \leq p^2 + pq = p(r-p) \leq r^2/4.$$

(b) Let $Y \in \mathbf{V}$ have the maximal rank p among the matrices in $\mathbf{V}$. Because $Y^2 = 0$, we may apply a similarity transform and assume that

$$Y = \begin{pmatrix} 0_p & I_p & 0 \\ 0_p & 0_p & 0 \\ 0 & 0 & 0_q \end{pmatrix}$$

with $2p+q=r$.

Let $Z \in \mathbf{V}$. Since $YZ = ZY = 0$, it follows that

$$Z = \begin{pmatrix} 0_p & Z_{12} & Z_{13} \\ 0_p & 0_p & 0 \\ 0 & Z_{32} & Z_{33} \end{pmatrix}. \tag{4.6}$$

We claim that $Z_{33} = 0$ when $q \geq 1$. Note that $tY + Z$ has rank at most p for all $t \in \mathbb{F}$. Let $G(t,j,k)$ be a submatrix of $\begin{pmatrix} tI_p + Z_{12} & Z_{13} \\ Z_{32} & Z_{33} \end{pmatrix}$ of size $p+1$ including the leading principal submatrix $tI_p + Z_{12}$ and the (j,k) entry, γ_{jk} say, of $Z_{3,3}$ as the $(p+1,p+1)$ entry. Then, $\det(G(t,j,k))$ is a polynomial with leading term $\gamma_{jk}t^p$, and $\det(G(t,j,k)) = 0$ for all $t \in \mathbb{F}$. Since $\mathbb{F}$ has more than $r/2 \geq p + 1/2$ elements, namely, $\mathbb{F}$ has at least $p+1$ elements, $\det(G(t,j,k))$ is the zero polynomial and $\gamma_{jk} = 0$. Because this is true for any entry γ_{jk} of Z_{33}, we see that $Z_{33} = 0_q$. Consequently, every Z in $\mathbf{V}$ has the form (4.6) with $Z_{33} = 0_q$ and $Z_{13}Z_{32} = 0_p$.

Let $\{r_1, \ldots r_{k_1}\}$ be a basis for the vector space spanned by the row vectors of matrices Z_{13} with all $Z \in \mathbf{V}$, and $\{c_1, \ldots, c_{k_2}\}$ be a basis for the vector space spanned by the column vectors of matrices in Z_{32} with all $Z \in \mathbf{V}$. Let $B_1 \in \mathbf{M}_{k_1,q}(\mathbb{F})$ have row vectors $r_1, \ldots, r_{k_1}$, and $B_2 \in \mathbf{M}_{q,k_2}(\mathbb{F})$ have column vectors $c_1, \ldots, c_{k_2}$. We can think of all r_i^{t} and all c_j as vectors in $\mathbb{F}^q$ with $r_i c_j = 0$. In other words, $B_1 B_2 = 0$. It follows that the nullity of the rank k_1 matrix B_1 is at least k_2, which is the rank of B_2. The rank–nullity theorem implies that $q - k_1 \geq k_2$.

For any $Z \in \mathbf{V}$, we have $Z_{13} = XB_1$ for some $X \in \mathbf{M}_{p,k_1}(\mathbb{F})$ and $Z_{32} = B_2 Y$ for some $Y \in \mathbf{M}_{k_2,q}(\mathbb{F})$. Hence, $Z \in \mathbf{V}$ has the asserted form. ∎

Our proof used ideas in [23], in which the structure of a vector subspace $\mathbf{W}$ of $\mathbf{M}_r(\mathbb{F})$ carrying trivial Jordan product is given. In particular, the following was obtained.

Proposition 4.6 ([23, Theorem 4])**.** *Let $\mathbf{W}$ be a vector subspace of $\mathbf{M}_r(\mathbb{F})$ consisting of square zero matrices. Suppose that $\mathbb{F}$ has more than $r/2$ elements, and all matrices in $\mathbf{W}$ have rank at most p. Then $\mathbf{W}$ has dimension at most $p(r-p) \leq r^2/4$.*

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let $\Phi : \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ be a real or complex linear map preserving double zero products. If $\Phi(I_n) = 0$, by Remark 3.8 we see that $\Phi(A)^2 = 0$ for all $A \in \mathbf{M}_n(\mathbb{F})$. It then follows from Proposition 4.6 that the range space $\mathbf{W} = \Phi(\mathbf{M}_n(\mathbb{F}))$ has dimension at most $r^2/4$.

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